TWO ENUMERATIVE PROBLEMS IN ALGEBRAIC GEOMETRY

by

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ABSTRACT

In this thesis we explore two enumerative problems from algebraic geometry, each with its own flavor.

First, we investigate Le Potier's Strange Duality for moduli spaces of sheaves on surfaces. Strange duality is a conjectural perfect pairing between spaces of sections of the determinant line bundles (also called theta line bundles) on moduli spaces with discrete invariants satisfying a suitable orthogonality condition. Our approach, inspired by [MO07], is to construct zero dimensional Quot schemes whose points correspond to dual bases for the two spaces of sections. We restrict our study to the case when one of the moduli spaces is the Hilbert scheme of points on a del Pezzo surface. We compute the expected cardinality of these Quot schemes using multiple point formulas [MR10]. These numbers are shown to be equal to the Euler characteristics of the theta line bundles, as computed by the universal power series of [EGL01]. We also investigate conditions under which we can prove the existence of suitable Quot schemes. As part of this, we prove a result of independent interest: a general sheaf on $\mathbb{P}^2$ of Euler characteristic at least 2 greater than its rank is globally generated.

We also attach a paper about tropical geometry. In this paper, we consider the question of when points in tropical affine space uniquely determine a tropical hypersurface. We introduce a notion of multiplicity of points so that this question may be meaningful even if some of the points coincide. We give a geometric/combinatorial way and a tropical linear-algebraic way to approach this question. First, given a fixed hypersurface, we show how one can determine whether points on the hypersurface determine it by looking at a corresponding marking of the dual complex. With a regularity condition on the dual complex and when the number of points is minimal, we show that our condition is equivalent to the connectedness of an appropriate subcomplex. Second, we introduce notions of nonsingularity of tropical matrices and solutions to tropical linear equations that take into account our notion of multiplicity and prove a Cramer's Rule type theorem relating them.

CONTENTS

LIST OF FIGURES

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CHAPTER 1

INTRODUCTION

Enumerative questions have long been an inspiring source of problems in algebraic geometry. In this thesis, we examine two different problems with an enumerative flavor.

First, we look at some special cases of Le Potier's "strange duality" for algebraic surfaces. Although the original statement involves a duality of vector spaces, the enumerative aspect shows up in our work as we count the points of certain finite Quot schemes that produce dual bases for these vector spaces. This thesis contains sections written by the author for the forthcoming paper [BJG16], some introductory material, and some discussion of details and points that did not make it into the paper.

Secondly, we examine some questions in tropical geometry questions about when points impose independent conditions on linear series of hypersurfaces. When the number of hypersurfaces is finite, then, as in the classical case, the number is always 0 or 1, so the interesting question is to ask when the number is finite. This thesis contains the full text of a paper by the author [Joh15] that has been submitted for publication.

CHAPTER 2

STRANGE DUALITY FOR DEL PEZZO SURFACES

This chapter contains details about the project on strange duality.

2.1 Preliminaries

Here we review some classical facts from algebraic geometry that are relevant to us.

2.1.1 Hilbert polynomial

Let $(X, \mathcal{O}_X(1))$ be a smooth polarized variety. Given a coherent sheaf F on X, one defines the Hilbert polynomial of F to be

$$P_F(k) = \chi(X, F(k)).$$

The Hilbert polynomial is constant in flat families, which makes it a natural choice for a discrete invariant in moduli problems.

The Hilbert polynomial may be computed from the chern character of F and a knowledge of the intersection theory of X by use of the Hirzebruch-Riemann-Roch theorem, which we make frequent use of.

Theorem 2.1.1 (Hirzebruch-Riemann-Roch, [Hir66])**.** *Let F be a coherent sheaf on X. Then*

$$\chi(F) = \int \mathrm{ch}(F) \cdot \mathrm{td}(X).$$

Here $\cdot$ is the intersection product on Chow (or cohomology) and $\int$ is taking the degree of the $\dim X$ part, and $\mathrm{td}(X)$ is the todd class, which is a certain polynomial combination of the chern classes of the tangent bundle of X.

From this theorem, one can see that the Hilbert polynomial is, in fact, a polynomial.

We will also make use of the *reduced Hilbert polynomial*. Let r be the rank of F. One then defines the reduced Hilbert polynomial to be

$$p_F(k) = P_F(k)/r.$$

2.1.2 Quot schemes

The Quot scheme is an important example of a moduli space. Given a vector bundle V, one wishes to parametrize the quotients of V with some fixed Hilbert polynomial P, up to an equivalence relation. Two quotients $f : V \to F$ and $f' : V \to F'$ are said to be equivalent if there is a isomorphism $\phi : F \to F'$ so that $\phi \circ f = f'$.

The Quot scheme answers the moduli problem posed by the functor which takes a base scheme B and returns the set of equivalence classes of flat families $\mathcal{F}$ on $B \times X$ with a surjective map $p_2^* V \to \mathcal{F}$ such that every fiber $\mathcal{F}|_{b \times X}$ has Hilbert polynomial P. (Here p_2 is the second projection $B \times X \to X$.)

Theorem 2.1.2 (Grothendieck, [Gro62])**.** *There is a projective scheme* $\mathrm{Quot}_P(V)$ *which represents the functor described above.*

Once a polarization is fixed, the Hilbert polynomial is determined by a chern character. Given a chern character f, we write $\mathrm{Quot}(V, f)$ for the associated Quot scheme.

2.1.3 Moduli spaces of sheaves

Next we recall some basic facts and notation about moduli spaces of sheaves.

The first thing one tries is to fix a Hilbert polynomial P and define the functor $\mathcal{M}_P$ from schemes to sets

$$\mathcal{M}_P(B) = \{\text{flat families over } B \text{ of sheaves on } X \text{ with Hilbert polynomial } P\}/\sim$$

and to ask whether this functor is representable. Here $\sim$ is the equivalence relation where two families are equivalent if they are isomorphic after tensoring with the pullback of a bundle on the base B.

Unfortunately, even after fixing P, the functor $\mathcal{M}_P$ can not be represented by a scheme of finite type. The solution is to restrict to families of the so-called semistable sheaves.

A torsion-free sheaf F is said to be stable (respectively, semistable) if, for all subsheaves $E \subset F$, one has

$$p_E < p_F \quad (\text{respectively, } p_E \leq p_F)$$

for the reduced Hilbert polynomials. Here, the ordering on polynomials is the lexicographic one, or, equivalently, one says $p_E < p_F$ if $p_E(k) < p_F(k)$ for sufficiently large k.

One can show that the set of semistable sheaves with fixed Hilbert polynomial is bounded, that is, there is a family with a base of finite type that contains all of these sheaves.

Unfortunately, a scheme that actually represents the functor is still too much to ask for. Instead, we get the notion of corepresentability:

Definition 2.1.3. A scheme M *corepresents* a functor $\mathcal{M}$ if there is a natural transformation

$$\eta : \mathcal{M} \to \mathrm{Hom}(-, M)$$

that is initial among schemes with natural transformations from $\mathcal{M}$, that is, given a scheme N and natural transformation $\nu : \mathcal{M} \to \mathrm{Hom}(-, N)$, then there exists a unique morphism $\phi : M \to N$ so that $\nu = \phi_* \circ \eta$.

The word *corepresents* here is a shortened form of coarsely represents (no duality is intended).

Notice that this notion is weaker than that of a coarse moduli space, since we do not require a bijection between closed points of M and isomorphism classes of the objects to be parameterized. Indeed, in general, this is not the case for moduli spaces of sheaves. M induces an coarser equivalence relation on isomorphism classes of sheaves (two sheaves are equivalent if they map to the same point in moduli). Fortunately, one can describe this equivalence relation somewhat explicitly.

Assume F is a semistable sheaf. Then F has a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = F$$

such that F_{i+1}/F_i is a stable sheaf. The factors that occur in the filtration are unique up to ordering. Such a filtration is called the Jordan-Holder filtration. Two semistable sheaves that have the same factors (ignoring the order) in their Jordan-Holder filtrations are called S-equivalent. Notice that the S-equivalence class of a stable sheaf consists of only the sheaf itself (up to isomorphism).

Theorem 2.1.4 ([Gie77],[Mar77],[Mar78],[Sim94]). *The functor $\mathcal{M}_P$ is corepresented by a projective scheme M_P. The equivalence relation induced by M_P is precisely S-equivalence.*

As we have noted, a chern character v, together with the polarization $\mathcal{O}_X(1)$, determines a Hilbert polynomial P. In what follows, we usually write $M(v)$ for the moduli space M_P.

2.1.4 Hilbert schemes of points on surfaces

One particular moduli space that we will be interested in is the Hilbert scheme of points on a surface. Given a surface S and positive integer n, the Hilbert scheme $S^{[n]}$ parameterizes zero-dimensional subschemes of length n. An open subset of the Hilbert scheme is parameterizing unordered n-tuples of points on S. Indeed, the Hilbert scheme has a birational morphism (called the *Hilbert-Chow morphism*) to the symmetric product $S^{(n)}$. In fact, it is a resolution of $S^{(n)}$:

Theorem 2.1.5 (Fogarty, [Fog86]). *$S^{[n]}$ is an irreducible smooth projective variety of dimension $2n$.*

In our paper, we are interested in the case when $X = S$ is a del Pezzo surface. Then the Picard group is discrete and the Hilbert scheme is a moduli space of sheaves on surfaces, that is, it is the moduli space of sheaves with chern character equal to $(1, 0, -n)$. (We write chern characters as $(\mathrm{rank}, \mathrm{ch}_1, \mathrm{ch}_2)$, where the last entry is understood to be the coefficient of the class of point.) Torsion-free sheaves of this chern character are precisely ideal sheaves of zero-dimensional length n subvarieties.

We next recall the standard description of the Picard group of $S^{[n]}$. Given a line bundle L on S, one constructs a line bundle L_n on $S^{[n]}$ as follows. First take $\bigotimes_{i=1}^n pr_i^*(L)$ on S^n, where pr_i is the ith projection $S^n \to S$ from the Cartesian product. This line bundle comes equipped with an action of the permuation group $\mathfrak{S}_n$, and hence descends to a line bundle on the symmetric product $S^{(n)}$, which can then be pulled back to $S^{[n]}$ via the Hilbert Chow morphism. For a divisor D, D_n can be thought of as the locus of subschemes whose support meets D.

This gives a map $\mathrm{Pic}(S) \to \mathrm{Pic}(S^{[n]})$, which induces an isomorphism

$$\mathrm{Pic}(S) \oplus \mathbb{Z}\left[\mathcal{O}_{S^{[n]}}\left(-\tfrac{B}{2}\right)\right] \cong \mathrm{Pic}(S^{[n]}), \tag{2.1}$$

where B is the exceptional divisor of the Hilbert-Chow morphism.

2.2 The general setup of strange duality

Given a variety X, the Chow ring (or cohomology ring) of X is endowed with a pairing

$$(e,f) := \int e^* \cdot f \cdot \mathrm{td}(X).$$

Here $(-)^*$ is the operation that negates the odd degree components so that, for example, a locally free sheaf E satisfies $\mathrm{ch}(E)^* = \mathrm{ch}(E^*)$. If $\mathrm{ch}(E) = e$ and $\mathrm{ch}(F) = f$, then one has

$$(e,f) = \chi(E,F) := \sum_{i=1}^{\dim X} (-1)^i \operatorname{Ext}^i(E,F) \tag{2.2}$$

Now let us assume that we have selected chern characters that are orthogonal with respect to this pairing, that is $(e,f) = 0$. One expects then that for general E and F, $\operatorname{Ext}^i(E,F) = 0$ for all i. In fact, in good cases, there will be a divisor $\Theta \subset \mathcal{M}(e) \times \mathcal{M}(f)$ supported on the locus

$$\{(E,F) : \operatorname{Hom}(E,F) \neq 0\}.$$

Fixing a sheaf E with $\mathrm{ch}(E) = e$, one can restrict Θ to obtain a divisor Θ_E in $\mathcal{M}(f)$ supported on the locus

$$\{F : \operatorname{Hom}(E,F) > 0\}.$$

If one assumes that there is a universal family $\mathcal{F}$ on $\mathcal{M}(f) \times S$, then one can define the theta divisor more precisely as

$$\Theta_E = -c_1(Rp_*(R\,\mathcal{H}om(q^*E, \mathcal{F}))). \tag{2.3}$$

(Here p and q are the projections from $\mathcal{M}(f) \times S$.) If there is not a universal family, one can use Theorem 8.15 in [HL10] to obtain a precise description.

Similarly, one obtains a divisor Θ_F on $\mathcal{M}(e)$.

Now we add the hypothesis that $X = S$ is a del Pezzo surface and hence has a discrete Picard group. In this case, the line bundles Θ_E and Θ_F depend only on the chern characters of e and f. The line bundle associated to Θ splits as box product

$$\Theta_E \boxtimes \Theta_F.$$

Hence, one obtains

$$H^0(\mathcal{M}(e) \times \mathcal{M}(f), \Theta) \cong H^0(\mathcal{M}(e), \Theta_F) \otimes H^0(\mathcal{M}(f), \Theta_E)$$

The section Θ then induces a morphism

$$SD : H^0(\mathcal{M}(f), \Theta_E) \to H^0(\mathcal{M}(e), \Theta_F)^*. \tag{2.4}$$

The strange duality conjecture then predicts that this should be an isomorphism.

Remark 2.2.1. One can also set this up in a more symmetric way by using the pairing

$$(e, f) = \int e \cdot f \cdot \mathrm{td}(X) = \chi(e \cdot f)$$

and considering the locus where $H^0(E \otimes F)$ has a section. We have chosen to do it with $\mathrm{Hom}(E, F)$ because it is more compatible with our quot scheme interpretation in Section 2.6.

Example 2.2.2. For an example where Θ_E is a divisor, consider $S = \mathbb{P}^2$, $E = \mathcal{O}_{\mathbb{P}^2}(-1)$ and $f = (1, 0, -3)$. Then $M(f) = (\mathbb{P}^2)^{[3]}$. The divisor Θ_E is precisely the locus of ideal sheaves $I_{p,q,r}$ where $\mathrm{Hom}(\mathcal{O}(-1), I_{p,q,r}) \neq 0$, which happens precisely when p, q, and r lie on a line, which is a divisor in $(\mathbb{P}^2)^{[3]}$.

Example 2.2.3. As an example of how Θ can fail to be a divisor, consider the case where S is a del Pezzo surface and $E = I_p$, an ideal sheaf of one point p, and $F = \mathcal{O}_S$, the trivial line bundle. The todd class on a del Pezzo is $1 - K/2 + [\mathrm{pt}]$, where K is the canonical divisor, so we quickly check that $\chi(E, F) = 0$. But $\mathrm{Hom}(I_p, \mathcal{O}) \neq 0$ for any p.

2.3 Strange duality with non-discrete Picard group

The discussion above applies to the case of del Pezzo surfaces which are the main subject of our paper. In order to adapt the setup to curves or abelian surfaces, some care is needed. The theta divisors Θ_E are not necessarily linearly equivalent for different E of the same chern class.

Let C be a smooth curve of genus g and for convenience let $\bar{g} = g - 1$. Following the notation of [MO07], we let $SU(r, \mathcal{O})$ be the moduli space of rank r vector bundles on a curve C with trivial determinant. This space has a theta divisor θ_r which may be viewed as the locus of points parametrizing bundles whose tensor product with a fixed line bundle of degree $\bar{g}$ has a section. Let $U(k, k\bar{g})$ be the moduli space of rank k bundles with degree $k\bar{g}$.

This space has a theta divisor Θ_k determined by the locus of points parametrizing bundles with a section. Then we have an isomorphism ([MO07])

$$SD : H^0(SU(r,\mathcal{O}),\theta_r^k)^* \to H^0(U(k,k\bar{g}),\Theta_k^r). \tag{2.5}$$

The interchanging of the r and k is sometimes called “the level rank duality.”

This classical version of strange duality for curves looks different from our general set in a couple ways. First, the theta divisors are factored as a power of a primitive theta divisor. Secondly, due to the presence of a nondiscrete Picard group it is necessary to fix the determinant on one side. However, it is possible to instead pull back corrections from the determinant map $\det : U(r,d) \to \mathrm{Pic}^d(C)$. This yields a more symmetric and (to us) appealing statement.

Such a statement can be extracted from [MO07], however it is not stated this way there.

Theorem 2.3.1. *Let C be a smooth curve of genus g. Fix a vector bundle E of rank r and degree d and F of rank r' and degree d' so that $\chi(E\otimes F)=0$. Further, pick a factorization of the cannonical bundle $K = C\otimes D$ into lines bundles of degree $\bar{g}$.*

Then, there is a SD isomorphism

$$H^0(U(r',d'),\Theta_E\otimes \det{}^*(\Theta_{D\otimes \det E^*})) \cong H^0(U(r,d),\Theta_F\otimes \det{}^*(\Theta_{C\otimes\det F^*}))^*$$

Proof. First, in order to match notation with [MO07], we make the following substitutions

$$r' \mapsto hr, \qquad d' \mapsto hd + hr\bar{g}$$
$$r \mapsto kr, \qquad d \mapsto -kd$$

One can check that choosing r and d (on the right hand side) relatively prime and h and k any positive integers will parametrize all pairs of orthogonal ranks and degrees.

Now, we indicate how to adapt the proof of [MO07] to prove this statement. Proceed as in Section 5 of [MO07]. However, don’t do the whole computation “as in Lemma 1,” but leave it as

$$\mathcal{O}(\Delta) = \Theta_{E\otimes M}\otimes \det{}^*(\Theta_{\det E^*\otimes(\det Q)^{h+k}\otimes L^*\otimes K})\boxtimes\Theta_{F\otimes M}\otimes\det{}^*\Theta_{\det F^*\otimes Q^{h+k}\otimes L}$$

on $U(hr,hd)\times U(kr,-kd)$.

Now, the proof in that paper shows that the strange duality holds for the spaces of sections of the two factors of this box product.

Now, apply the isomorphism $-\otimes M : U(hr, hd) \to U(hr, hd + hr\bar{g})$ to the first factor. Also, notice that this map is compatible with the map $-\otimes M^{hr} : Pic^{hd} \to Pic^{hd+hr\bar{g}}$. Under this, the first factor becomes

$$\Theta_E \otimes \det{}^*(\Theta_{\det E^* \otimes (\det Q)^{h+k} \otimes L^* \otimes K \otimes M^{-hr}}).$$

On the other side, let $F' = F \otimes M$, so $(\det F')^* = \det F^* \otimes M^{-hr}$ and we have

$$\Theta_{F'} \otimes \det{}^* \Theta_{(\det F')^* \otimes M^{hr} \otimes Q^{h+k} \otimes L}$$

Now, to obtain the statement in our theorem (with the renaming of the ranks and degrees), rename F' to F. Pick any reference bundle Q and line bundle M, then set $L = D \otimes (\det Q)^{-h-k} \otimes M^{-hr}$. $\square$

For abelian surfaces, the situation is even more complicated. Let $M^+(v)$ be the moduli of sheaves with chern character v and fixed determinant and $M^-(v)$ be the moduli of sheaves with fixed determinant of the Fourier-Mukai transform (with kernel equal to the Poincare bundle). Further, let $K(v)$ be the moduli of sheaves with both determinant and determinant of the Fourier-Mukai transform fixed. Let v and w be orthogonal chern characters. Then, one has the following strange duality maps

$$H^0(K(v), \Theta_w)^* \to H^0(M(w), \Theta_v)$$
$$H^0(M^+(v), \Theta_w)^* \to H^0(M^+(w), \Theta_v)$$
$$H^0(M^-(v), \Theta_w)^* \to H^0(M^-(w), \Theta_v)$$

These are numerically supported by a computation of their dimensions in [MO08] and proved in many cases in [MO14] and [BMOY14].

One is led to ask whether there is a more symmetric statement that mirrors that of Theorem 2.3.1, which can be restricted to recover the results above. According to private communication with the second author of [MO14], there is numerical evidence that supports this, but no statement or proof is currently available.

2.4 The determinant bundle for Hilbert schemes

We now consider the case when $f = (1, 0, -n)$ is the set of invariants of an ideal sheaf so that $\mathcal{M}(f)$ is the Hilbert scheme $S^{[n]}$. In this case, we can identify the determinant line bundle Θ_E explicitly.

One can parametrize a general chern character orthogonal to $(1, 0, -n)$ as

$$e = (r, -L, (n-1)r + \frac{L.K}{2}). \tag{2.6}$$

Here K is the class on the canonical divisor on S. We write the first chern character as $-L$, since in the future we will require L to have some positivity properties.

The class of the determinant line bundle induced by a sheaf E is

$$\Theta_{E^*} = -c_1(Rp_*(R\,\mathcal{H}om(q^*E, \mathcal{I}_{\mathcal{Z}}))).$$

Next, we wish to determine which line bundle this is in terms of the isomorphism (2.1) in Section 2.1.4. We first introduce the operation $(-)^{[n]}$ which takes a sheaf on S and gives a sheaf on $S^{[n]}$. This operation is the Fourier-Mukai transform with kernel equal to the structure sheaf $\mathcal{O}_{\mathcal{Z}}$ of the universal subscheme on $S \times S^{[n]}$. More explicitly, if p and q are the first and second projections, respectively, $F^{[n]} = q_*(p^*F \otimes \mathcal{O}_{\mathcal{Z}})$.

Let G have rank r and determinant L. (For our notation, we view L as a divisor.) Then, in [Wan13], one has the formula

$$\det(G^{[n]}) = \mathcal{O}_{S^{[n]}}(L_n - \frac{r}{2}B). \tag{2.7}$$

Here B is the class of of the exceptional divisor of the Hilbert-Chow morphism $S^{(n)} \to S^{[n]}$.

Starting with the exact sequence

$$0 \to I_{\mathcal{Z}} \to \mathcal{O} \to \mathcal{O}_{\mathcal{Z}} \to 0$$

on $S \times S^{[n]}$, one then tensors by (the locally free) sheaf p^*E^*, and then applies Rq_*, obtaining an exact triangle

$$Rq_*(\mathcal{I}_{\mathcal{Z}} \otimes p^*E^*) \to H^0(S, E^*) \otimes \mathcal{O}_{S^{[n]}} \to (E^*)^{[n]} \to \cdots$$

The determinant bundle Θ_E is the negative of the determinant of the first term in this complex, and hence is equal to the determinant of $(E^*)^{[n]}$, given in (2.7).

2.4.1 Euler characteristics of determinant line bundles

Next, we wish to compute the Euler characteristics of the determinant line bundles (2.7). A technique to do this is provided by the following theorem:

Theorem 2.4.1 ([EGL01], Theorem 5.3)**.** *For any surface S,*

$$\sum_{n\geq 0} \chi\left(\mathcal{O}_{S^{[n]}}(L_n - r\tfrac{B}{2})\right) z^n = g_r(z)^{\chi(L)} \cdot f_r(z)^{\frac{1}{2}\chi(\mathcal{O}_S)} \cdot A_r(z)^{L.K_S - \frac{1}{2}K_S^2} \cdot B_r(z)^{K_S^2}. \tag{2.8}$$

where $A_r(Z)$, $B_r(Z)$, $f_r(z)$, $g_r(z)$ are power series in z depending only on r, and

$$f_r(z) := \sum_{k\geq 0} \binom{(1-r^2)(k-1)}{k} z^k$$

$$g_r(z) := \sum_{k\geq 0} \frac{1}{1-(r^2-1)k} \binom{1-(r^2-1)k}{k} z^k.$$

Notice that what is called E in [EGL01] we call $-B/2$.

Explicit formulas for $A_r(z)$ and $B_r(z)$ are not known, but as explained in [EGL01], one can use the localization techniques of [ES96] and [ES87] to determine the first few terms of the power series on the left hand side for $\mathbb{P}^2$ and any r and L. Then, substituting in strategic choices of L, one can solve for the first few terms of $A_r(z)$ and $B_r(z)$. In [EGL01], $A_r(z)$ and $B_r(z)$ are computed up to order 5. This was not sufficient for our purposes, so we implemented the suggested computations in Sage [The16]. We will explain this method in more detail in Section 2.5. See Section B for some results.

2.5 Localization computations

Our goal is to compute the Euler characteristic $\chi((\mathbb{P}^2)^{[n]}, \mathcal{O}((kH)_n - \frac{r}{2}B))$ as a function of k, n, and r. By the Hirzebruch-Riemann-Roch formula, this can be accomplished by computing the integral of a certain polynomial in classes of $(kH)_n$, B and chern classes of the tangent bundle of $(\mathbb{P}^2)^{[n]}$, namely

$$\chi((\mathbb{P}^2)^{[n]}, \mathcal{O}((kH)_n - \frac{r}{2}B)) = \int \exp((kH)_n - \frac{r}{2}B) \cdot \mathrm{td}(T_{(\mathbb{P}^{[n]})}).$$

Localization is a powerful technique which allows one to compute integrals of polynomials in the chern classes of equivariant bundles on varieties with a $\mathbb{C}^\times$ action.

First we recall some notation. Let X be a variety with a $\mathbb{C}^\times$ action with isolated fixed points. Let F be the set of fixed points. For example, one could take $X = \mathbb{P}^2$ and $\mathbb{C}^\times$

actions of the form $\lambda \cdot (x_0 : x_1 : x_2) = (x_0 : \lambda^a x_1 : \lambda^b x_2)$. The fixed points would then be, in homogeneous coordinates, $(1 : 0 : 0)$, $(0 : 1 : 0)$, and $(0 : 0 : 1)$. The case most interesting to us is the case of the Hilbert scheme $X = (\mathbb{P}^2)^{[n]}$ with action induced by an action on $\mathbb{P}^2$.

Let $\mathcal{E}$ be a rank k equivariant vector bundle on X, that is, one such that the $\mathbb{C}^\times$ action on X lifts to a linear action on the fibers of $\mathcal{E}$.

If $x \in X$ is a fixed point of the action, then the fiber $\mathcal{E}_x$ is a representation of $\mathbb{C}^\times$. Such representations decompose as a sum of one-dimensional representations, each determined by its weight. Let $\tau_1(\mathcal{E}, x), \ldots, \tau_k(\mathcal{E}, x)$ be these weights, and let $\sigma_1(\mathcal{E}, x), \ldots, \sigma_k(\mathcal{E}, x)$ be the elementary symmetric functions in the $\tau_i(\mathcal{E}, x)$.

In [ES96], it is explained that cohomology classes on X can be viewed as functions on the fixed point set. This identification takes the product in cohomology to the obvious product on functions.

Hence, in order to compute top products of chern classes of $\mathcal{E}$, we need first a method to associate a function to a chern class of $\mathcal{E}$, and second, given a function representing a zero cycle, we need to be able to find its degree.

These are supplied by the following theorem:

Theorem 2.5.1 (Theorem 2.2, [ES96]). *We have the following:*

- *The function associated to $c_k(\mathcal{E})$ is $x \mapsto \sigma_k(\mathcal{E}, x)$.*
- *If f is a function representing a zero cycle, then the degree of the zero cycle is*

$$\sum_{x \in F} \frac{f(x)}{\sigma_n(x)}.$$

In order to accomplish our computation, it is necessary to understand the fixed points of $\mathbb{C}^\times$ on $(\mathbb{P}^2)^{[n]}$. In [ES96], it is explained how the fixed points of $(\mathbb{P}^2)^{[n]}$ correspond to triparitions of n. The authors also show how one can determine the representation of the maximal torus acting on the fibers of $\mathcal{O}(H_n)$ and the tangent bundle, and from information given there, one can work out the representation on the fibers of $\mathcal{O}(-\frac{1}{2}B)$. Then, given a subgroup $\mathbb{C}^\times$ of the maximal torus, one can substitute to obtain the weights of the $\mathbb{C}^\times$ action on these fibers.

We now have all the ingredients. We make one more useful observation: as a corollary of Theorem 2.5.1, if a polynomial in the chern classes of $\mathcal{E}$ is given as a symmetric function of the chern roots, one obtains the corresponding function on fixed points by simply substituting in the weights τ_i for the chern roots. The todd class is given this way. Hence, once the weights for the tangent bundle are known, it is quick to find the function on the fixed point set corresponding the the todd class.

The computations described here were implemented by the author in Sage [The16]. See Section A.2 for the code and Section B for the resulting series $A_r(Z)$ and $B_r(z)$.

2.6 Quot schemes and the main theorem

The technique we use to approach strange duality is inspired by the proof in [MO07] for curves. We outline the strategy here.

We first make a general observation. Given sections s_i for $i = 1, \ldots, n$ of a line bundle L, one can show that these sections are independent by finding points $x_i \in X$ so that the square matrix $(s_i(x_j))$ is nonsingular.

Let e and f be orthogonal chern characters with respect to the pairing (2.2). Then let $v = e + f$ and V be a general sheaf with $\mathrm{ch}(V) = v$. Consider the Quot scheme $\mathrm{Quot}(V, f)$. The kernel will have chern character e, and the tangent space at a point parameterizing an exact sequence

$$0 \to E \to V \to F \to 0$$

will be $\mathrm{Hom}(E, F)$. This is expected to be zero by the condition $(e, f) = 0$. That is, we expect $\mathrm{Quot}(V, e)$ to be a finite set. Let $\{E_i\}$ and $\{F_i\}$ be the collections of subsheaves and quotients of a fixed V obtained this way.

Next, we apply the general observation at the beginning of this section to the sections $\Theta_{F_i} \in H^0(\mathcal{M}(e), \Theta_e)$ and the points $[E_i] \in \mathcal{M}(e)$. Recall that Θ_{F_i} is the locus of sheaves E with $\mathrm{Hom}(E, F_i) \neq 0$. Hence we see that $[E_i] \notin \Theta_{F_i}$ (as the tangent space is assumed to be 0) but $[E_j] \in \Theta_{F_i}$ for $j \neq i$ since there is a nonzero map $E_j \to V \to F_i$. In other words, the matrix $\Theta_{F_j}([E_i])$ is diagonal with nonzero entries on the diagonal. Similar arguments apply to Θ_{E_i} and $[F_j]$. In fact, we see that these sections correspond to each other (up to scaling) under SD (2.4).

In [MO07], where strange duality is proved for curves, a suitable V is constructed, its quotients counted, and then a dimension formula for both sides of SD concludes the proof.

In our paper, we restrict to the case when f is the invariants of an ideal sheaf of n points. We prove the following:

Theorem 2.6.1. *Let* $f = (1,0,-n)$ *and* $e = (r, -L, (n-1)r + \frac{L.K}{2}$ *with* $r \geq 1$ *and* L *sufficiently positive. Let* V *be a general vector bundle with* $\mathrm{ch}(V) = e+f$. *Then the expected cardinality of the Quot scheme* $\mathrm{Quot}(V,f)$ *is equal to the Euler characteristic* $\chi(M(f), \Theta_e) = \chi(S^{[n]}, \mathcal{O}(L_n - \frac{r}{2}B)$.

Of course, it would be desirable to compute the dimension of $H^0(\mathcal{M}(e), \Theta_{(1,0,-n)})$, but this seems out of reach in all but a few cases.

The qualifier *expected* in the Theorem has reference to the fact that we must assume that certain maps in our construction are *admissible* from the point of view of multiple-point formulas. Unfortunately, there seems to be no useful way to check whether a specific map is admissible.

However, we do have an additional result where we find some examples where we can check that the Quot scheme is in fact finite.

Theorem 2.6.2. *Let* $1 \leq n \leq 7$ *and* λ *sufficiently large. Further assume that* λ *is odd and* 3 *does not divide both* λ *and* $n+1$.

Let V *be a general stable sheaf on* $\mathbb{P}^2$ *with chern character* $(3, -\lambda H, n-2-\frac{3\lambda}{2})$. *Then the Quot scheme* $\mathrm{Quot}(V,(1,0,-n))$ *is a finite, reduced set, and each quotient is an ideal sheaf of* n *reduced points.*

As part of this check, we are able to prove a new result that is of independent interest.

Theorem 2.6.3. *Let* V *be a vector bundle with* $c_1(V) > 0$ *and* $\chi(V) \geq 2 + \mathrm{rank}(V)$. *Then the general* V *in moduli is globally generated.*

2.7 Multiple-point computations

In this section we explain how to obtain the expected cardinality of a Quot scheme whose quotients are ideal sheaves by the use of multiple-point formulas.

2.7.1 First motivating example: $r = 2$, $n = 2$

In the notation of (2.6), we set $r = 2$ and $n = 2$. Using Theorem 2.4.1 (although this can be done with less technology), one can compute

$$\chi(\mathcal{O}_S(L_2 - B)) = \binom{\chi(L)}{2} - 3\chi(L) - L.K + 3. \tag{2.9}$$

(here $\chi(L) = \chi(\mathcal{O}_S(L))$). We wish to compare this formula to the expected cardinality of the Quot scheme of I_2 quotients of the rank 3 vector bundle V. Such a quotient is the same as a section of V^* vanishing at 2 points.

In this case we have

$$\mathrm{ch}(V^*) = (3, L, \frac{L.K}{2})$$

and

$$\chi(V^*) = 3.$$

We assume $H^0(V^*) = 3$. By the properties of chern classes, there is a curve $C \subset S$ of class L where the 3 sections drop rank. Let us assume C is smooth. There is a map

$$f : C \to P(H^0(V^*)) \cong \mathbb{P}^2$$

where $p \in C$ maps to a section that vanishes at p. (We do not expect the sections to drop rank by more than one, so let us assume this is well defined.)

The point of this construction is now that a double point of $f(C)$ corresponds to a section of V^* vanishing at 2 points. Hence, our next task is to compute the number of singularities of $f(C)$.

First, we compute the degree of $f(C)$ in $\mathbb{P}(H^0(V^*)) \cong \mathbb{P}^2$. If we ask for the number of points of intersection with a general line, we are asking for the number of points where a dimension 2 subspace of $H^0(V^*)$ drops rank, which by the properties of chern classes is $c_2(V^*)$. We also know by the adjunction formula on S that genus of C is $\frac{c.(c+K)}{2} + 1$, so we see that the image of C has a number of singularities equal to the difference between the genus of a smooth plane curve of degree $c_2(V^*)$ and the genus of C

$$\binom{c_2(V^*) - 1}{2} - \left(\frac{c.(c + K)}{2} + 1\right). \tag{2.10}$$

We assume that each of these singularities corresponds to a double point of $f(C)$. One can now make the substitution $c_2(V^*) = \frac{c_1(V^*)^2}{2} - \mathrm{ch}_2(V^*) = \frac{c^2}{2} - \frac{c.K}{2}$ and $\frac{1}{2}c(c-K)+1 = \chi(c)$ to see that (2.10) is equal to

$$\binom{\chi(c)-2}{2} - (\chi(c) + c.K).$$

Expanding this, we see that it matches (2.9), as desired.

It is possible to make this more rigorous. In [BJG16], we show how one can arrange for C to be smooth, and the map $f : C \to \mathbb{P}^2$ can be arranged (by choosing V appropriately) to be the composition of an embedding in a larger projective space followed by a general projection. Hence $f(C)$ will have only double points as singularities.

2.7.2 Counting ideal sheaf quotients more generally

How does one generalize the computation in Section 2.7.1? In general, we have

$$\chi(V^*) = n(r-1) + 1.$$

We assume that this is equal to $h^0(S, V^*)$. If this number is at least two greater than the rank of V, then we expect V^* to be generated by global sections. We define G as the kernel of the global sections map:

$$0 \to G \to H^0(S, V^*) \otimes \mathcal{O}_S \to V^* \to 0. \tag{2.11}$$

Consider now the projective bundle $\mathbb{P}(G^*)$ of lines in the fibers of G (we use the contravariant version of $\mathbb{P}$ here). We view a point of this bundle over a point $p \in S$ as a section of V^* that vanishes at p. We then define a map to projective space

$$f : \mathbb{P}(G^*) \to \mathbb{P}(H^0(S, V^*)^*)$$

by "forgetting" the base. More precisely, one applies $\mathbb{P}$ to the dual of the first map in (2.11) to obtain $\mathbb{P}(G^*) \to \mathbb{P}(H^0(S, V^*)^*) \times S$ and then projects onto the first factor.

We then interpret the n-fold point locus of this map as the set of sections of V^* vanishing at n points.

2.7.3 Computation of multiple points

In this section we explain how to count the n-fold points described in the previous section, generalizing the genus-degree comparison in Section 2.7.1.

An n-fold point in the target of a map $X \to Y$ is a point $y \in Y$ so that $f^{-1}(y)$ has precisely n points. The codimension of the the n-fold point locus in Y is $n(\dim X - \dim Y)$. In particular, if the codimension of the map is $\frac{1}{n}\dim Y$, one expects there to be finitely many n-fold points.

As a sanity check for our strategy in Section 2.7.2, we check that the dimension of $\mathbb{P}(G)$ is

$$\dim S + \operatorname{rank} G - 1 = 1 + \chi(V^*) - \operatorname{rank} V.$$

On the other hand, the target space has dimension $\chi(V^*) - 1$. Hence, this map has codimension $\operatorname{rank} V - 2 = r - 1$. Since $\chi(V^*) - 1 = n(r-1)$, this is consistent with our expectation that there are finitely many multiple points.

If we are looking at double points ($n = 2$), and if f is an immersion, then the number of these can be computed (in terms of the chern classes of the tangent bundle of X and the pull back by f of the chern classes of the tangent bundle of Y) by the Herbert-Ronga formula [Ron80], [Her81].

Extending this, Kleiman in [Kle81] gives a multiple-point formula for any n and maps of corank 1, that is, maps that fail to be an immersion, but the induced map on tangent spaces has kernel of dimension at most one at any point.

Unfortunately, in the cases we are interested in, the maps can be as bad as corank two (that is, the tangent directions along the surface may be collapsed), so Kleiman's formulas are not sufficient.

The current state of the art is in [MR10], where a general formula for any corank map is proved for $n \leq 7$. The formula is conjectured to hold for $n \geq 8$, but this is unknown. It would be interesting to provide evidence or counter-evidence for this conjecture by attempting to extend Theorem 2.6.1 to $n = 8$. Unfortunately, the formulas become computationally infeasible when $n = 8$, so we are unable to provide a check for this conjecture by comparing it to (2.8).

We give a quick overview of these formulas. One can derive from results of [MR10] the following:

Proposition 2.7.1. *Assume $f\colon X \to Y$ is an admissible morphism of codimension ℓ between complex manifolds. Then for $n \leq 7$, the n-fold locus has class*

$$y_n = \frac{1}{n}\sum_{i=0}^{n-1}(-1)^i f_*(R_{A_i}(\ell-1))\, y_{n-1-i} \ \in H^{2n\ell}(Y,\mathbb{C}).$$

Notice that if the codimension is $\frac{1}{n}\dim Y$, then this is a class in $H^{2\dim Y}(Y,\mathbb{C})$, so y_n may be viewed as a number.

Here the $R_{A_i}(\ell-1)$ are the Thom polynomials of the A_i singularities. These can be computed via formula (7.26) in [BS12].

To do this, let $RC(q) = 1 + c_1 q + c_2 q^2 + c_3 q^3 + \cdots$ be the generating series of the relative chern classes of the map (that is, the chern classes of T_X/f^*T_Y, or, still equivalently, the series obtained as the quotient of the series of the chern classes of T_X by the pullback of the series for T_Y). Then the formula says:

$$R_{A_i}(\ell-1) = \mathrm{Res}_{\mathbf{z}=\infty} \frac{(-1)^i \prod_{m<l}(z_m - z_l)\widehat{Q_i}(z_1,\ldots,z_d)}{\prod_{l=1}^{i}\prod_{m=1}^{l-1}\prod_{r=1}^{\min(m,l-m)}(z_m + z_r - z_l)} \prod_{l=1}^{i} \mathrm{RC}\left(\frac{1}{z_l}\right) z_l^{\ell-1} dz_l$$

Notice that in [BS12], R_{A_d} is called $\mathrm{eP}[\Theta_d]$, with the codimension suppressed from the notation. The result of this computation is a polynomial in the chern classes of the relative tangent bundle of the map.

In our case, one can check (see [BJG16] for details) that the chern classes with degree greater than $\ell + 4$ always vanish, which simplifies the computation somewhat.

The $\widehat{Q_i}$ are auxiliary polynomials that can be computed as described in [BS12]. This seems to be the limiting step in the computation. $\widehat{Q_6}$ is a degree 13 polynomial in 6 variables. One needs to compute the primary decomposition of a ideal generated by a large combinatorially defined set of relations to extract it. $\widehat{Q_7}$ is a degree 22 polynomial in 7 variables, and the ideal that it is hidden in seems to be too large to handle.

The explicit results of the computation are included in Appendix B.

2.8 Bridgeland stability

We first give a brief description of stability conditions as a background for the following sections.

The bounded derived category $D^b(\mathcal{A})$ of an abelian category $\mathcal{A}$ is constructed by first taking the homotopy category of $\mathcal{A}$, that is, one takes the category whose objects are bounded complexes in $\mathcal{A}$ and whose morphisms are homotopy classes of chain maps. On the homotopy category, one then performs an operation called *localization*, in which quasiisomorphisms are given inverses. The result is the bounded derived category $D^b(\mathcal{A})$, which has the structure not of an abelian category, but what is know as a *triangulated* category. A triangulated category is equipped with a shift functor, commonly written as [1] acting on the right. In the case of $D^b(\mathcal{A})$, this shift functor comes from the functor on the homotopy category that shifts the indices of complexes.

One is then led to ask how the abelian category $\mathcal{A}$ is related to the triangluated category $D^b(\mathcal{A})$. The answer is that it is the heart of a t-structure on $D^b(\mathcal{A})$. The heart of a t-structure is (in particular) an abelian subcategory of the triangulated category. A triangulated category has many possible hearts of t-structures.

Given a heart of a triangulated category, one can obtain a new one by *tilting by torsion pair.* A *torsion pair* for an abelian category is a pair $(\mathcal{T}, \mathcal{F})$ of full additive subcategories with the property that

$$\mathrm{Hom}(T, F) = 0$$

for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Further, we require that every object E in the abelian category fit into an exact sequence

$$0 \to T \to E \to F \to 0.$$

For example, in the abelian category of coherent sheaves on X, one could take $\mathcal{T}$ to be the set of torsion sheaves and $\mathcal{F}$ to be the set of torsion-free sheaves.

If the abelian category is the heart of a t-structure in a triangulated category, one then defines the tilt by $(\mathcal{T}, \mathcal{F})$ to be the abelian subcategory generated by $\mathcal{T}$ and $\mathcal{F}[1]$. One can check that this will again be a heart.

A *stability function* on an abelian category $\mathcal{A}$ is an additive homomorphism Z from the Grothendeik group $K(A)$ into the semiopen upper half-plane

$$\mathbb{H} = \{z = a + bi : b > 0 \text{ or } a < 0\}.$$

The phase φ_E of an object E is the unique $0 \leq \varphi < 1$ so that

$$Z(E) = re^{i\varphi/\pi}.$$

The prototypical example of this is the function $-\deg + i\,\mathrm{rank}$ on the category of coherent sheaves on a curve.

An object E is called semistable (with respect to Z) if for all subobjects F one has

$$\varphi_F \leq \varphi_E.$$

A stability function is said to satisfy the Harder-Narasimhan property if every object admits a filtration by semistable objects, that is, for all F there is a filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = F$$

with F_{i+1}/F_i semistable.

Finally, we define a stability condition on a triangulated category to be a t-structure together with a stability function satisfying the Harder-Narasimham property on the heart. A stability condition on a variety is a stability condition on the derived category of the category of coherent sheaves on the variety.

Given a stability condition, one can attempt to construct moduli spaces of semistable objects. Here is one result that is relevant to us:

Theorem 2.8.1 (see Proposition 3.3 in [BMW13])**.** *Let S be a del Pezzo surface and Z be a stability condition. Then the moduli space of semistable objects is a Mori dream space. In particular, it is a projective variety.*

The manifold of stability conditions has a *wall and chamber* structure. That is, there are walls which partition it into chambers so that in each chamber the moduli space of semistable objects is the same. When one crosses a wall (by changing the stability condition), one obtains a new birational model. For example, in [BC13], the relationship between the minimal model program for $\mathbb{P}^{[n]}$ and the variation of stability conditions is explored.

2.8.1 Determinant bundles and Bridgeland models

In this section we will explain how Bridgeland moduli spaces are relevant to strange duality.

Let (S, H) be a polarized del Pezzo surface. The even cohomology

$$H^{2\bullet}(S, \mathbb{Q}) = H^0(S, \mathbb{Q}) \oplus H^2(S, \mathbb{Q}) \oplus H^4(S, \mathbb{Q})$$

is equipped with a nondegenerate Poincaré pairing:

$$\langle \alpha, \beta \rangle = \int \alpha.\beta$$

Here the dot is the intersection product on cohomology. We also make use of the operator $(-)^*$, which changes the sign of the $H^2(S,\mathbb{Q})$ component. It has the property that $\mathrm{ch}(E^*) = \mathrm{ch}(E)^*$.

Given an element $\alpha = \alpha_0 + \alpha_1 + \alpha_2 \in H^{2\bullet}(S,\mathbb{Q})$ satisfying $\alpha_1^2 > 2\alpha_0\alpha_2$, one can produce a stability condition with central charge (see [Ber15]):

$$Z_\alpha(G) = -\langle \alpha, \mathrm{ch}(G)\rangle + \sqrt{-1}\,\langle \alpha.H, \mathrm{ch}(G)\rangle \tag{2.12}$$

The heart of this stability condition is a tilt on the category of coherent sheaves, with $\mathcal{T}$ being the extension closure of the collection of semistable sheaves G such that $\langle \alpha.H, \mathrm{ch}(G)\rangle \geq 0$.

For a fixed chern character g and a stability condition determined by α as above, one obtains a moduli space of Bridgeland semistable objects $M^\alpha(g)$.

For any stability condition σ, Bayer and Macri produce in [BM14] a numerical divisor class $\ell_{\sigma,g}$ on $\mathcal{M}^\sigma(g)$ and show that it is nef. This class is given by

$$\ell_{\sigma,g}(C) = \Im\left(\frac{-Z_\sigma(\Phi_{\mathcal{G}}(\mathcal{O}_C))}{Z_\sigma(g)}\right). \tag{2.13}$$

Here $\phi_{\mathcal{G}}$ is the Fourier-Mukai transform with kernel $\mathcal{G}$, the universal family on $S \times \mathcal{M}^\sigma(g)$, and $\Im(-)$ means to take the imaginary part.

By Theorem 2.8.1 it is known that the Bridgeland moduli spaces of semistable objects with respect to this stability condition is projective, so in fact this divisor class is ample.

Now we return to our strange duality set up, with e and f orthogonal with respect to (2.2). If we set $\alpha = e^*.\,\mathrm{td}(S)$, one obtains a stability condition producing a moduli space $M^\alpha(f)$. We observe that $Z_\alpha(f)$ is purely imaginary. Similarly, one can use $\alpha = f^*.\,\mathrm{td}(S)^*$ to obtain a moduli space $M^\alpha(e)$.

On these moduli spaces, it turns out that the numerical divisor class (2.13) is the same (up to a rational scalar) as the determinant line bundle. To see this, we first note that it is sufficient to consider a family $\mathcal{F}$ of semistable sheaves of chern character f on a curve C. Let p and q be the first and second projections from $S \times C$, respectively. Then

$$\ell_{\alpha,f}(C) = -\frac{\langle \alpha, Rp_*(\mathcal{F})\rangle}{\langle \alpha.H, f\rangle}$$

One sees (by Hirzebruch-Riemann–Roch) that the numerator is equal to $\chi(E^* \otimes Rp_*(\mathcal{F}))$. Since Euler characteristic is invariant under pushforward, using the projection formula this is equal to $\chi(p^*(E^*)\otimes\mathcal{F})$. Then, pushing forward by q, this is equal to $\chi(Rq_*(p^*(E^*)\otimes\mathcal{F}))$. We wish to compare this to the intersection of C with the theta divisor (2.3). This intersection number is just $-\deg Rq_*(p^*E^*\otimes\mathcal{F})$. This is a rank 0 sheaf on a curve, hence its degree is equal to its Euler characteristic. Hence, the $\ell_{\alpha,f}(C)$ matches with $\Theta_e.C$ up to the positive constant $\langle\alpha.H, f\rangle$.

This gives a preferred model of the moduli space to use for strange duality. In some situations, this can make analyzing the determinant bundle more tractable, for example, when $\alpha = e^*.\operatorname{td}(S)$ determines a stability condition at the last wall or in the last chamber when the moduli space is simpler.

2.8.2 A rank one example

In this section we restrict ourselves to a case where the strange duality can be made explicit on the level of representations, making use of the previous section. We take $S = \mathbb{P}^2$ and $f = (1, 0, -1)$. We let $L = \lambda H$, where H is the hyperplane class on $\mathbb{P}^2$. The moduli space $M_{\mathbb{P}^2}(f)$ is just $\mathbb{P}^2$.

The determinant line bundle on this moduli space is $\mathcal{O}_{\mathbb{P}^2}(\lambda)$. Its sections are $\operatorname{Sym}^\lambda W$, where $W = H^0(\mathcal{O}_{\mathbb{P}^2}(1))$.

It works out better in this case to work with the moduli space $M(e^*)$ instead of $M(e)$. We have

$$e^* = (r, \lambda, -\frac{3}{2}\lambda).$$

Then, the theta divisor on the product $M(e^*)\times\mathbb{P}^2$ is

$$\Theta = \{(E^*, p) : H^0(\mathbb{P}^2, E^*\otimes I_p) \neq 0\}$$

One can run the algorithm described in [CHW14] and see that a general sheaf E^* with such invariants has a resolution of the form

$$0 \to \mathcal{O}(-2)^{\oplus\lambda} \to \mathcal{O}(-1)^{\oplus\lambda}\oplus\mathcal{O}^{\oplus r} \to E^* \to 0 \tag{2.14}$$

In an appropriate tilt of the category of coherent sheaves in its derived category, one can view this as an exact sequence: The stability condition we wish to consider for the moduli of E^* is induced (as in Section 2.8.1) by

$$\alpha = (1, 0, -1)\cdot\operatorname{td}(\mathbb{P}^2) = (1, \frac{3}{2}, 0).$$

Notice that $\langle \alpha, \operatorname{ch} \mathcal{O} \rangle = 0$, so we can view the resolution above as the Jordan-Holder filtration of E^*:

$$\mathcal{O}^{\oplus r} \to E^* \to [\mathcal{O}(-2)^{\oplus \lambda} \to \mathcal{O}(-1)^{\oplus \lambda}].. \tag{2.15}$$

Hence the moduli space for this stability condition is the Kronecker space which parameterizes maps

$$\mathcal{O}(-2)^{\oplus \lambda} \to \mathcal{O}(-1)^{\oplus \lambda}. \tag{2.16}$$

That is, the data of the extension class in (2.15) is lost in S-equivalence. However, by the result of [BM14] discussed in Section 2.8.1, we know that the determinant bundle induced by $(1, 0, -1)$ is ample on this space, so we have lost no information about the sections of of the determinant bundle.

A map (2.16) is the same as a map

$$\mathbb{C}^\lambda \to \mathbb{C}^\lambda \otimes W, \tag{2.17}$$

or, it can be thought of as a $\lambda \times \lambda$ matrix with entries in W (i.e., the entries are linear forms on $\mathbb{P}^2$).

We can see this space as a quotient of a Grassmanian

$$\operatorname{Gr}(\lambda, \mathbb{C}^\lambda \otimes W)//SL(\lambda)$$

Next, we argue that the determinant bundle descends from the Plucker line bundle on this Grassmannian. Let I_p be the ideal sheaf of a fixed point $p \in \mathbb{P}^2$. We wish to analyze the locus where $H^0(E^* \otimes I_p) \neq 0$. Tensoring the sequence (2.14) with I_p preserves exactness, so we see that $H^0(E^* \otimes I_p)$ may be identified with the kernel of

$$H^1(I_p(-1))^{\oplus \lambda} \to H^1(I_p(-2))^{\oplus \lambda}.$$

This map may be identified with

$$H^0(\mathcal{O}_p(-1))^{\oplus \lambda} \to H^0(\mathcal{O}_p(-2))^{\oplus \lambda} \tag{2.18}$$

via twists of the standard exact sequence for I_p. Of course, his map is just a map $\mathbb{C}^\lambda \to \mathbb{C}^\lambda$. One can obtain this map by evaluating the map (2.16) at p. To determine if this map has a non-trivial kernel, one can take the determinant of the matrix (2.17) and evaluate it at p. So we wish to find the locus of points of the Grassmannian where this happens.

The target space of the Plücker embedding is $\mathbb{P}\left(\bigwedge^\lambda(\mathbb{C}^\lambda \otimes W)\right)$. The point p determines a map $W \to \mathbb{C}$. This gives a linear map $\bigwedge^\lambda(\mathbb{C}^\lambda \otimes W) \to \bigwedge^\lambda \mathbb{C}^\lambda$ which corresponds to evaluating the determinant of the matrix (2.17) at p. Hence the locus of points where this determinant vanishes, and hence (2.18) has a kernel and $H^0(E^* \otimes I_p) > 0$, is a hyperplane section of the Plucker embedding, as claimed.

The invariant sections of the Plucker line bundle are

$$\left(\bigwedge^{\lambda}\left(\mathbb{C}^\lambda \otimes W^*\right)\right)^{SL(\lambda)}.$$

Now, this can be decomposed (as a $GL(\lambda) \times GL(W^*)$ module) as

$$\bigoplus_\sigma S_\sigma(\mathbb{C}^\lambda) \otimes S_{\sigma^\dagger}(W^*)$$

where the sum is over paritions σ of λ, and S_σ is the Schur functor associated to σ. The only summand that has invariants is when $\sigma = (1, \ldots, 1)$ so S_σ is $\bigwedge^t$ and $\sigma^\dagger = (\lambda)$ so $S_{\sigma^\dagger}$ is Sym^λ. So the invariants are

$$\bigwedge^{\lambda}\left(\mathbb{C}^\lambda\right) \otimes \mathrm{Sym}^\lambda W^* \cong \mathrm{Sym}^\lambda W^*$$

which we see as naturally dual to the sections of λH on $\mathbb{P}^2$.

2.8.3 A rank two example

In this section we give another interesting example where the discussion in Section 2.8.1 gives us an advantage. In general, it seems difficult to compute the sections of determinant bundles on higher rank moduli spaces, but here we can compute both sides on the level of representations.

Consider the sheaves F with invariants $(2, 3, -\frac{5}{2})$ pairing with I_2 sheaves on $\mathbb{P}^2$. (Here we are working with the more symmetric setup indicated in Remark 2.2.1.) By Formula 2.8, we see that the determinant bundle on $\mathbb{P}^{[2]}$ has 27 sections. Now, using the resolution by exceptional sheaves in [CHW14], we see that a general sheaf with such invariants fits into an exact triangle (this is an exact sequence in the heart)

$$T(-1) \to F \to [\mathcal{O}(-2) \to \mathcal{O}]$$

(here T is the tangent bundle on $\mathbb{P}^2$). When the stability function is chosen so that $T(-1)$ and $[\mathcal{O}(-2) \to \mathcal{O}]$ (and thus also F) have the same phase, then this becomes the Jordan-Holder filtration of F, and the moduli space is collapsed on a $\mathbb{P}^5$ parametrizing maps

$\mathcal{O}(-2) \to \mathcal{O}$. If one moves the stability condition from this so that $T(-1)$ has phase slightly larger than F, then all of the Fs will be unstable, so the moduli space is empty. Hence, we call the ray of stability conditions where the phases are equal *the last wall.* On the other side of the last wall (in the last chamber), the Fs are stable, and different extension classes count as different sheaves. Hence the moduli space in the last chamber is a projective bundle over $\mathbb{P}^5$ with 11-dimensional fibers $\mathbb{P}(\mathrm{Ext}^1([\mathcal{O}(-2) \to \mathcal{O}], T(-1)))$.

Unfortunately we do not have the tools to check that the stability condition induced by I_2 on $\mathcal{M}(f)$ is close enough to the last wall to land in the last chamber. Qualitatively it seems "close", however, so we will assume that the theta line bundle is ample on this model of the moduli space.

We now wish to determine which vector bundle (with fibers as described above) on $\mathbb{P}^5$ gives this last chamber moduli space. Notice that the universal map on $\mathbb{P}^5 \times \mathbb{P}^2$ is a map $\mathcal{O}(-1,-2) \to \mathcal{O}$ given in projective coordinates by $a_{11}x^2 + a_{12}xy + \dots$. Hence, the bundle we want is

$$\begin{aligned}\mathcal{E}xt^1([\mathcal{O}(-1,-2) \to \mathcal{O}], q^*T(-1)) &= R^1p_*([\mathcal{O} \to \mathcal{O}(1,2)] \otimes q^*T(-1)) \\ &= R^1p_*([q^*T(-1) \to \mathcal{O}(1) \boxtimes T(1)]).\end{aligned}$$

Now, notice that both terms of the complex are p_*-acyclic, that is, the higher cohomology of $T(1)$ and $T(-1)$ vanishes, hence

$$\begin{aligned}Rp_*([q^*T(-1) \to \mathcal{O}(1) \boxtimes T(1)]) &= [p_*(q^*T(-1)) \to p_*(\mathcal{O}(1) \boxtimes T(1))] \\ &= [H^0(T(-1)) \otimes \mathcal{O} \to H^0(T(1)) \otimes \mathcal{O}(1)] \\ &\cong [\mathcal{O}^{\oplus 3} \to \mathcal{O}(1)^{\oplus 15}].\end{aligned}$$

Hence the vector bundle $\mathcal{E}$ that we want to projectivize to form the moduli space is the cokernel of the complex in the last line above.

Next, we search for a line bundle on $\mathbb{P}(\mathcal{E})$ that has 27 sections. We find that

$$H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathcal{E}}(-1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^5}(2)) = H^0(\mathbb{P}^5, \mathcal{E}^* \otimes \mathcal{O}(2)) = H^0(\mathbb{P}^5, \ker(\mathcal{O}(1)^{\oplus 15} \to \mathcal{O}(2)^{\oplus 3})). \tag{2.19}$$

If we assume $H^1(\mathcal{E}^*(2)) = 0$, then this has dimension $90 - 63 = 27$, as desired.

In fact, with some work, one can check the duality on the level of representations of PGL_3. We let $W = H^0(\mathbb{P}^2, \mathcal{O}(1))$ be the standard representation. Writing (2.19) more carefully, the sections on the bundle over $\mathbb{P}^5 = \mathbb{P}(S^2(W)^*)$ are the sections of the kernel of

$$\mathcal{O}_{\mathbb{P}^5}(1) \otimes H^0(T_{\mathbb{P}^2}(1))^* \to \mathcal{O}_{\mathbb{P}^5}(2) \otimes H^0(T_{\mathbb{P}^2}(-1))^* \to 0$$

hence they are the kernel of the map of vector spaces

$$S^2W^* \otimes H^0(T_{\mathbb{P}^2}(1))^* \to S^2S^2W^* \otimes H^0(T_{\mathbb{P}^2}(-1))^*$$

From the Euler sequence on $\mathbb{P}^2$, one can identify the representation $H^0(T(d))$ as $S^{d+2,d+1}(W)$ (for $d \geq -1$), where here S is the Schur functor corresponding the the partition indicated in the superscripts. One can now use a program such as Schur Group Theory Software [Wyb14] to compute the plethysms and tensor products above. The source is

$$S^{5,4}(W) \oplus S^{4,2}(W) \oplus S^{3,3}(W) \oplus S^3(W) \oplus S^{2,1}(W)$$

while the target is

$$S^{5,4}(W) \oplus S^{3,3}(W) \oplus S^3(W) \oplus S^{2,1}(W)$$

If we assume that this map is surjective, the kernel is the irreducible, self dual, 27 dimensional representation $S^{4,2}(W)$.

Now we wish to think about the determinant line bundle on $(\mathbb{P}^2)^{[2]}$. To do so, recall that $(\mathbb{P}^2)^{[2]}$ may be thought of as the projective bundle corresponding to $\Omega(1)$ on $(\mathbb{P}^2)^*$. (Each fiber is the Hilbert scheme of 2 points on the corresponding line.) It seems difficult to carefully identify the determinant line bundle from this point of view, but one can check that taking $\mathcal{O}_{\Omega(1)}(1)$ and twisting by $\mathcal{O}_{(\mathbb{P}^2)^*}(4)$ from the base gives 27 sections, so we assume that this is the correct bundle.

Hence we wish to compute the representation given by $H^0(S^2(\Omega_{(\mathbb{P}^2)^*}(1))(4))$. First, we resolve $S^2(\Omega_{(\mathbb{P}^2)^*}(1))$ as

$$0 \to \bigwedge^2 \left(\Omega_{(\mathbb{P}^2)^*}(1)\right) \to \Omega_{(\mathbb{P}^2)^*}(1)^{\otimes 2} \to S^2\left(\Omega_{(\mathbb{P}^2)^*}(1)\right) \to 0.$$

After twisting this sequence by $\mathcal{O}(4)$ and noting that $\bigwedge^2 \Omega = \omega = \mathcal{O}(-3)$, we see that it is sufficient to know $H^0(\mathcal{O}(3)) = S^3(W^*) = S^{3,3}(W)$ and $H^0(\Omega(1) \otimes \Omega(5))$. To compute the latter, we tensor the Euler sequence for $\Omega(1)$ by $\Omega(5)$. We can check (from the Euler sequence again) that $\Omega_{(\mathbb{P}^2)^*}(d) = S^{d-1,d-2}$ (remember $H^0(\mathcal{O}_{(\mathbb{P}^2)^*}(1)) = W^*$), and again using Schur, we obtain

$$H^0(\Omega(1) \otimes \Omega(1)) = S^{4,2}(W) \oplus S^{3,3}(W),$$

from which it follows that $H^0(S^2(\Omega_{(\mathbb{P}^2)^*})(1))(4)) = S^{4,2}(W)$ as desired.

From this example, one might be led to guess that representations of this form are irreducible. Unfortunately when $r = 2$, $n = 2$, and $\lambda = 4$ the dimension is 75, which is not the dimension of an irreducible PGL_3 representation. Even if one insists that the invariants are primitive, $r = 2$, $n = 2$, and $\lambda = 7$ has dimension 546, which also cannot be irreducible.

In future work, we would like to extend our range of examples like this by using a better strategy. One can view $\mathbb{P}^2$ as a homogeneous space. Then, homogeneous vector bundles on $\mathbb{P}^2$ will correspond to certain representations, which we can compute tensor products and plethysms with, and then compute an induced representations to find the sections.

2.9 Finite Quot schemes exist

As evidence that our computations proving Theorem 2.6.1 are meaningful, in this section we prove Theorem 2.6.2 and exhibit some sheaves V on $\mathbb{P}^2$ that in fact have finitely many I_n quotients.

This section is in a preliminary state, and since the original submission, some errors have been found. In particular, Proposition 2.9.8 is false as stated, and the discussion in the last paragraph of the section is based on an unproved conjecture. We refer the reader to our forthcoming paper [BJG16] for improvements and corrections.

Since we are working on $\mathbb{P}^2$, we write a chern character as a triple of numbers, where the middle one is understood to be the coefficient of the hyperplane class.

We restate Theorem 2.6.2 here for convenience:

Theorem 2.9.1. *Let $1 \leq n \leq 7$ and λ sufficient large. Further assume that λ is odd and 3 does not divide both λ and $n+1$.*

Let V be a general stable sheaf on $\mathbb{P}^2$ with chern character $(3, -\lambda, n-2-\frac{3\lambda}{2})$. Then the Quot scheme $\mathrm{Quot}(V, (1, 0, -n))$ is a finite, reduced set, and each quotient is an ideal sheaf of n reduced points.

We would like to conclude that in these cases, the multiple point computations actually compute the length of the Quot scheme. Assuming this, we have:

Corollary 2.9.2. *With the hypotheses in Theorem 2.9.1, the strange duality morphism*

$$SD : H^0((\mathbb{P}^2)^{[n]}, \Theta_e) \to H^0(\mathcal{M}(e), \Theta_{I_n})^*$$

is injective.

2.9.1 Proof

First, we sketch a general setup, and then we will restrict to the special case described in the hypothesis of Theorem 2.9.1.

Let e, f be orthogonal Chern characters of vector bundles on a del Pezzo surface S, that is, $\chi(e, f) = 0$. Let $v = e + f$ and assume that the moduli space $\mathcal{M}(v)$ is nonempty of the expected dimension and $\mathcal{M}(v)$ (or a dense open subset of it) supports a relative Quot scheme which we call $\mathcal{Q} \to \mathcal{M}(v)$. The fiber of $\mathcal{Q}$ over a point of $\mathcal{M}(v)$ parameterizing a sheaf V is the Quot scheme $\mathrm{Quot}(V, f)$ of quotients of V with invariants f. For example, if $\mathcal{M}(v)$ admits a universal family $\mathcal{V}$, then $\mathcal{Q} = \mathrm{Quot}_{\mathcal{M}(v)\times S}(\mathcal{V}, p_2^* f)$. Now, consider the product of moduli spaces $\mathcal{M}(e) \times \mathcal{M}(f)$. For appropriate choices of invariants, $\chi(f, e)$ will be negative and there will be an open subset $W \subset \mathcal{M}(e) \times \mathcal{M}(f)$ where the dimension of $\mathrm{Ext}^1(F, E)$ is constant and equal to $-\chi(f, e)$. These vector spaces will assemble into a projective bundle which we call $\mathbb{P}(\mathcal{E}xt^1(\mathcal{F}, \mathcal{E}))$. Within this space there is an open subset U consisting of short exact sequences

$$0 \to E \to V \to F \to 0 \tag{2.20}$$

such that V is stable. This open subset has an obvious inclusion ψ to $\mathcal{Q}$.

$$\begin{tikzcd}
\mathcal{Q} \arrow[d, "\pi"] & U \arrow[l, hook', "\psi"'] \arrow[r, hook] & \mathbb{P}(\mathcal{E}xt^1(\mathcal{F}, \mathcal{E})) \arrow[d] & \\
\mathcal{M}(v) & & W \arrow[r, hook] & \mathcal{M}(e) \times \mathcal{M}(f)
\end{tikzcd}$$

One can compute:

Lemma 2.9.3. *In the notation and assumptions above,*

$$\dim \mathbb{P}(\mathcal{E}xt^1(\mathcal{F}, \mathcal{E})) = \dim \mathcal{M}(v).$$

Proof. Recall the the dimension of the moduli space $\mathcal{M}(e)$ is $1-\chi(e,e)$.

The dimension of $\mathbb{P}(\mathcal{E}xt^1(\mathcal{F},\mathcal{E}))$ is the dimension of the base plus the dimension of the fiber. Above, we assumed that $\dim \mathbb{P}(\mathrm{Ext}^1(F,E)) = -\chi(f,e)-1$. Hence the dimension is

$$1-\chi(e,e)+1-\chi(f,f)-\chi(f,e)-1$$

while the dimension of $\mathcal{M}(v)$ is

$$1-\chi(v,v) = 1-\chi(e+f,e+f) = 1-\chi(e,e)-\chi(f,f)-\chi(e,f)-\chi(f,e)$$

and the result follows. □

We would like to use this dimension count to conclude that π is generically qausifinite and then obtain our result. Here is a more careful statement of what we need to check.

Lemma 2.9.4. *Let e and f be orthogonal chern characters and let $v = e+f$. Assume that:*

1. *$M(v)$, $M(e)$, and $M(f)$ are nonempty of the expected dimension.*
2. *The universal Quot scheme $\mathcal{Q}$ as described above exists.*
3. *The universal sheaves $\mathcal{E}$ and $\mathcal{F}$ on $M(e)$ and $M(f)$, respectively, exist.*
4. *A general $V \in \mathcal{M}(v)$ has no short exact sequences (2.20) with E or F unstable.*
5. *A general V has no sequences (2.20) so that $\mathrm{Ext}^1(E,F) > -\chi(e,f)$.*
6. *A general V has a quotient with chern character e.*

Then the general $V \in \mathcal{M}(v)$ has finitely many sequences (2.20), and the pairs (E,F) occurring in such sequences are general.

Proof. Assumptions (1), (2), and (3) are that the constructions above make sense. Then (4) and (5) imply that the general fiber of π is contained in U, and (6) says that π is dominant. Stability is an open condition in families, so U must be open in the irreducible $\mathbb{P}(\mathcal{E}xt^1(\mathcal{F},\mathcal{E}))$ and by (4) it is also nonempty, so by Lemma 2.9.3 U has the same dimension as $M(v)$. A dominant map of reasonable (e.g., finite type over a field) schemes of the same dimension is generically quasifinite, so the result follows.

The assertion that the pairs (E,F) are general is meant to mean that there is no closed subset of $M(e)\times M(f)$ so that the general fiber of π has a quotient in the closed subset. That this is true is clear since the locus of such sequences in U is of smaller dimension than $M(v)$. $\square$

Checking the conditions of Lemma 2.9.4 for the cases in the hypothesis of Theorem 2.9.1 is enough to prove the Theorem, except for the statement about the reducedness of the Quot scheme. For that, one observes that since e and f are candidates for strange duality, the locus $\Theta \subset M(e)\times M(f)$ is a divisor. Hence, by the dimension count, it cannot be the case that a general V has a sequence (2.20) with $T_{[F]}\operatorname{Quot}(V,f)=\operatorname{Hom}(E,F)\neq 0$.

We now restrict the conditions in the hypothesis of Theorem 2.9.1: $S=\mathbb{P}^2$ and $f=\operatorname{ch}(I_n)=(1,0,-n)$ and rank of v equal to 3. A general chern character e satisfying $\chi(e,f)=0$ is written as

$$e=(2,-\lambda,2(n-1)-\frac{3\lambda}{2}).$$

and thus

$$v=e+f=(3,-\lambda,n-2-\frac{3\lambda}{2}).$$

First, from [LP97], we recall that there is a quantity called the discriminant $\Delta(\xi)$ of a chern character ξ, given by the formula

$$\Delta(\xi):=\frac{1}{2}\mu(\xi)^2-\frac{\operatorname{ch}_2(\xi)}{r(\xi)}$$

where r is the rank, and $\mu(\xi)=\mathrm{c}_1(\xi)/r(\xi)$ is the slope. In [LP97], a function δ is constructed and the following theorem is proved.

Theorem 2.9.5. *There exists a positive dimensional moduli space $\mathcal{M}(\xi)$ if and only if $\chi(\xi)$ and $c_1(\xi)$ are integral and $\Delta(v)>\delta(\mu)$. In this case $\mathcal{M}(\xi)$ is a normal, irreducible, factorial projective variety of dimension $1-\chi(\xi,\xi)$.*

The function δ has a complicated fractal-like structure, however, it is bounded above by 1, so the hypothesis of this theorem is clearly satisfied for v and e when $\lambda\gg 0$. The moduli space for $(1,0,-n)$ is the Hilbert scheme $(\mathbb{P}^2)^{[n]}$, so condition (1) is checked.

Next, to check (2) and (3), we prove:

Lemma 2.9.6. *Assume that λ is odd and 3 does not divide both $n+1$ and λ. Then the universal families on $M(e)$ and $M(v)$ exist.*

Proof. According to Le Potier in [LP97], if the rank, first chern class, and euler characteristic are relatively prime, then the universal family exists. One computes

$$\chi(e) = -3\lambda + 2n$$
$$\chi(v) = -3\lambda + n + 1.$$

Notice that λ being odd is necessary and sufficient to make the relevant invariants of e relatively prime. If 3 does not divide $n+1$, then $\mathrm{rank}(v) = 3$ and $\chi(v)$ are relative prime. If 3 does not divide λ, then $\mathrm{c}_1(V) = -\lambda$ and $\mathrm{rank}(v) = 3$ are relatively prime. □

Next, we recall a useful result.

Theorem 2.9.7 (Göttsche, Hirschowitz [GH98])**.** *Suppose g is a chern character on $\mathbb{P}^2$ so that $\chi(g) < 0$ and $\mu(g) > -3$. Then the locus*

$$\left\{[G] : H^0(G) > 0\right\} \subset \mathcal{M}(g)$$

has codimension at least 2.

We need a Lemma that strengthens Theorem 2.9.7 in a special case that is of interest to us. To prove this lemma, we need the following observation, which we state without proof.

Proposition 2.9.8. *Fix positive numbers d, ℓ, and k so that $\ell + k > \chi(\mathcal{O}_{\mathbb{P}^2}(d))$. Then the locus in $(\mathbb{P}^2)^{[\ell]}$*

$$\{[Z] : H^1(I_Z(d)) \geq -\chi(I_Z(d) + k\}$$

has codimension at least k in $(\mathbb{P}^2)^{[\ell]}$.

Lemma 2.9.9. *Let $g = (2, c, d)$ be a chern character so that that $\mathcal{M}(g)$ has the expected dimension $c^2 - 4d - 3$. Let m be a positive integer.*

Consider the locus T_m of points $[G] \in M(g)$ such that G is locally free and has at least m sections.

Then T_m is either empty or has codimension at least $m - \chi(g)$ in $M(g)$.

Proof. First, consider the case where general section of G does not drop rank on a curve. Such a $[G] \in T_m$ fits into an $m-1$ dimensional family of exact sequences

$$0 \to \mathcal{O} \to G \to I_Z(c) \to 0$$

with Z the ideal sheaf of a zero-dimensional subscheme of length $\frac{c^2}{2} - d$. (We here recall that by Theorem 2.9.5, a necessary condition for the moduli of G to have the expected dimension is $\frac{c^2}{8} - \frac{d}{2} \geq 0$, so this length is positive.)

Next we note that

$$\mathrm{Ext}^1(I_Z(c), \mathcal{O}) = H^1(I_Z(c) \otimes K)^*$$

Notice that the dimension of this is equal to the failure of Z to impose independent conditions on curves of degree $c-3$. Hence, for generic Z, this number should be $|Z| - \chi(\mathcal{O}(c-3)) = \frac{3}{2}c - d - 1$. This number may jump up for special Z, but by Proposition 2.9.8, we see that the locus in the Hilbert scheme $(\mathbb{P}^2)^{[|Z|]}$ where it jumps by k has codimension at least k. In the case that $|Z| < \chi(\mathcal{O}(c-3))$, only special Z will have nontrivial extensions, and this number bounds the codimension of such Z. Hence we see that the dimension of T_m is at most the dimension of the moduli of Z plus the expected (projective) dimension of the space of extensions (possibly negative) minus $m-1$:

$$3|Z| - \chi(\mathcal{O}(c-3)) - 1 - (m-1) = \frac{3}{2}c^2 - 3d - \binom{c-1}{2} - m = c^2 + \frac{3}{2}c - 3d - 1 - m$$

This gives a codimension of $-\frac{3}{2}c - d - 2 + m$. On the other hand, $\chi(g) = \frac{3}{2}c + d + 2$. We have concluded the proof for the case when the general section of G does not drop rank on a curve.

Consider next the image H of the global sections map $H^0(G) \otimes \mathcal{O} \to G$. We consider several cases. First, if H has rank 1, then it must be of the form $I_Z(e)$ for some non-negative integer e and zero-dimensional subscheme Z.

If $e = 0$, then actually $m = 1$ and the section does not drop rank on a curve, so the first case applies.

So assume $e > 0$. Since G is stable we have $e \leq \frac{c}{2}$, and we also know that $\chi(e) \geq m$. Now, we see that $G(-e)$ has at least one section. One computes that

$$\chi(G(-e)) = \chi(G) + e^2 - 3e - ce$$

By induction the locus of such sheaves has codimension $1-\chi(G(-e)) = 1-\chi(G)-e^2+3e+ce$. Hence we will be done if we show that $1-e^2+3e+ce \geq m$. Using $c \geq 2e$, we obtain

$$1-e^2+3e+ce \geq e^2+3e+1 \geq \frac{1}{2}e^2+\frac{3}{2}e+1 = \chi(e) \geq m$$

as desired.

Finally, consider the case when H is rank 2. The cokernel of $H \to G$ is a torsion sheaf T, with the codimension 1 support equal to a (possibly reducible) curve C. We claim that either a general section of G does not drop rank on a curve, or all sections of G drop rank on a common curve. Now, one can deduce from the Porteous Theorem that a general section of a globally generated, rank 2, torsion-free sheaf on a surface drops rank only on points, hence the cokernel I of $\mathcal{O} \to H'$ is torsion free, so we obtain a diagram

$$\begin{tikzcd}
 & \mathcal{O} \arrow[r, equal] \arrow[d] & \mathcal{O} \arrow[d] & & \\
0 \arrow[r] & H' \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & T \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & T \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

The extension K has rank greater than 1 only along C and possibly some points. Hence if the general section of G drops rank on a curve, it can only do so on a component of C. Since C has only finitely many components, there must be somewhere dense subset of the sections of G that drops rank on one of these components, call it C'. Such a subset would necessarily contain a basis, so every section vanishes on C'. Then we see that $G(-C')$ has m sections. Since $G(-C')$ has smaller Euler characteristic, we are done by induction. (The base case is when G has first chern character negative, in which case a semistable G has no sections.) $\square$

We are now ready to check the first half of condition (4).

Lemma 2.9.10. *For $\lambda \gg 0$, every subsheaf E of a general stable sheaf V is stable.*

Proof. If E is not stable, then as it has rank two, it must have a subsheaf that is a line bundle, say $\mathcal{O}(d)$, with $d \geq -\frac{\lambda}{2}$. This line bundle would destabilize V unless $d < -\frac{\lambda}{3}$. This line bundle gives a section of $V(-d)$. Let $x = \frac{d}{\lambda}$, so $-\frac{1}{2} \leq x < -\frac{1}{3}$. One then computes

$$\begin{aligned}\chi(V(-d)) &= \frac{3}{2}d^2 + \lambda d - 3\lambda + \frac{9}{2}d + 1 \\ &= (\frac{3}{2}x^2 + x)\lambda^2 - 3\lambda + \frac{9}{2}x\lambda + 1\end{aligned}$$

The roots of $\frac{3}{2}x^2 + x$ are 0 and $-\frac{2}{3}$, hence the leading coefficient of $\chi(V(-d))$ (viewed as a polynomial in λ) is negative for any choice of d, so for $\lambda \gg 0$, we have $\chi(V(-d)) < 0$. We also have that the slope of $V(-d)$ is $-\frac{\lambda}{3} - d > 0 > -3$, so Theorem 2.9.7 applies and we conclude that V is special. □

Finally, we address the stability of the quotients. Recall that on $\mathbb{P}^2$, any rank one torsion-free sheaf with vanishing first chern class is an ideal sheaf of a zero dimensional subscheme.

Lemma 2.9.11. *Suppose $\lambda \gg 0$. Then the general V has all quotients of chern character $(1, 0, -n)$ torsion free.*

Proof. First, we note that if there is torsion supported in one dimension, then composing with the map to the reflexive hull will produce a nonzero map from V to $\mathcal{O}(-1)$, and hence a section of $V^*(-1)$. However one computes that $\chi(V^*(-1)) = nr - n - \lambda - r$, which is negative for $\lambda \gg 0$. Then Theorem 2.9.7 applies and contradicts the generality of V.

It remains to consider quotients $V \to F$ such that F has zero dimensional torsion. Each such quotient can be obtained by starting with a locally free E' with chern character equal to $\mathrm{ch}(E) - (0, 0, k)$ and an ideal sheaf I_{n+k} of a length $n + k$ zero dimensional subscheme, then picking an extension V of these two, and then picking a quotient T of E' that is a zero-dimensional subscheme of length k. This is summarized in Figure 2.1. We wish to find an upper bound on the dimension of the space V occurring this way.

One quickly checks

$$\dim S^{[n+k]} = \dim S^{[n]} + 2k$$

$$\dim \mathcal{M}(e') = \dim \mathcal{M}(e) - 4k$$

$$\chi(I_{n+k}, E') = \chi(I_n, e) + k$$

We see that $\mathrm{ext}^1(I_{n+k}, E') = -\chi(I_{n+k}, E') + \mathrm{hom}(E', I_{n+k} \otimes K)$, so have

$$\mathrm{ext}^1(I_{n+k}, E') = -\chi(I_n, e) - k + \mathrm{hom}(E', I_{n+k} \otimes K).$$

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & & T \arrow[d] & \\
0 \arrow[r] & E \arrow[r] \arrow[d] & V \arrow[r] \arrow[d, "="] & F \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E' \arrow[r] \arrow[d] & V \arrow[r] & I_{n+k} \arrow[r] \arrow[d] & 0 \\
 & T \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}$$

Figure 2.1. Bad V

Let us fix ℓ and restrict our attention on the set of diagrams as in Figure 2.1 with $\hom(E', I_{n+k} \otimes K) = \ell$.

Recall from Lemma 2.9.3 that the dimension of $M(v)$ is equal to $\dim M(e) + \dim S^{[n]} - \chi(I_n, e)$. Hence adding the dimension counts above together (and subtracting 1 for the scaling of the extensions) yields an upper bound for the dimension of the space of such V as

$$\dim S^{[n]} + \dim M(e') + \mathrm{ext}^1(I_{n+k}, E') - 1 \tag{2.21}$$

$$= \dim M(v) + 2k - 4k - k + \ell - 1 \tag{2.22}$$

$$= \dim M(v) - 3k - 1 + \ell \tag{2.23}$$

Now, we see that $\mathrm{Hom}(E', I_{n+k}) \subset \mathrm{Hom}(E', K) = H^0((E')^* \otimes K)$. Also, we see that $\chi((E')^* \otimes K) = \chi(E \otimes K) + k$. Hence by Lemma 2.9.9 the locus in $M(e')$ where $\hom(E', I_{n+k} \otimes K) \geq \ell$ has codimension at least $\ell - \chi(E \otimes K) + k$. We may therefore subtract this number from (2.23) to obtain

$$\dim \mathcal{M}(v) - 2k - 1 + \chi(E^* \otimes K)$$

Since $\chi(E^* \otimes K) = -3\lambda + 2n < 0$, we are done. $\square$

The next lemma proves condition (5) of Lemma 2.9.4.

Lemma 2.9.12. *For $\lambda \gg 0$ and general V, any sequence* (2.20) *has* $\mathrm{Ext}^1(I_n, E) = -\chi(I_n, E)$.

Proof. By Lemma 2.9.10 we may assume E is stable, and so $\mathrm{Hom}(I_n, E) = 0$.

Next, $\mathrm{Ext}^2(I_n, E) = \mathrm{Hom}(E, I_n \otimes K) \subset \mathrm{Hom}(E, K) = H^0(E^* \otimes K)$. Now one checks that $\chi(E^* \otimes K) = -3\lambda + 2n$, so Lemma 2.9.9 applies and so we see that the locus of $[E]$ where $\mathrm{Ext}^2(I_n, E)$ jumps by k is codimension (much) greater than k. Hence, the dimension of the space of sequences

$$0 \to E \to V \to I_n \to 0$$

where $\mathrm{Ext}^1(I_n, E) > -\chi(I_n, E)$ has dimension less than that of $\mathbb{P}(\mathcal{E}xt^1(\mathcal{I}_n, \mathcal{E}))$, and hence dimension less than $\mathcal{M}(v)$ by Lemma 2.9.3. $\square$

It remains to check condition (6), that is, we need to know that a general V has at least one quotient with invariants I_n. This follows from our multiple-point computations: if the number obtained from the multiple-point formula is positive, then the number of multiple points must either be the correct number or be infinite. As discussed in Section 2.7, multiple points correspond to ideal sheaf quotients of V.

2.9.2 Global generation

In order for the multiple-point setup to make sense, we need V^* to be globally generated. Our final task then is to show that a general stable $V*$ is globally generated. We only state the result in this thesis. For the proof, see [BJG16].

Theorem 2.9.13. *Let $\xi = (r, \lambda, d)$ be a Chern character such that $r \geq 1$, $\lambda \geq 1$, and $\chi(\xi) \geq r + 2$. Then general sheaves in $M(\xi)$ are globally generated.*

CHAPTER 3

DETERMINING TROPICAL HYPERSURFACES

This is a reproduction (with some corrections) of the paper [Joh15].

3.1 Introduction

We first recall the classical situation. Given a lattice polytope $\Delta \subset \mathbb{R}^n$, let $\mathbb{Z}(\Delta) = \Delta \cap \mathbb{Z}^n$ and $|\Delta| = |\mathbb{Z}(\Delta)|$. One can consider the linear system of hypersurfaces in $(\mathbb{C}^*)^n$ given by Laurent polynomials of the form

$$\sum_{I=(i_1,\ldots,i_n)\in\mathbb{Z}(\Delta)} a_I x_1^{i_1} \cdots x_n^{i_n}.$$

One can also view these hypersurfaces as lying in the toric variety corresponding to Δ.

Requiring the hypersurface to pass through a given point in $(\mathbb{C}^*)^n$ imposes a linear condition on the coefficients a_I, so we see that we require $|\Delta|-1$ points in general position to uniquely determine the hypersurface (up to uniform scaling of the coefficients). Given an explicit set of $|\Delta|-1$ points, one can check whether these points are general by verifying that the $(|\Delta|-1) \times |\Delta|$ matrix formed by evaluating monomials at the points is full rank. To compute the coefficients, one solves the associated homogeneous linear system. For example, one could accomplish both tasks via Cramer's rule by computing the maximal minors of the matrix.

3.1.1 Tropical hypersurfaces and higher codimension conditions

Recall that a tropical polynomial

$$f = \bigoplus_{I=(i_1,\ldots,i_n)\in\mathbb{Z}(\Delta)} a_I \odot x_1^{\odot i_1} \odot \cdots \odot x_n^{\odot i_n}$$

defines a subset $V(f) \subset \mathbb{T}^n$ called a tropical hypersurface by the condition $\mathbf{x} \in V(f)$ if

$$\min_I \left\{ a_I + \sum_{k=1}^{n} i_k x_k \right\}$$

is achieved by at least two choices of I. (The tropical preliminaries needed in this paper will be reviewed in more detail in Section 3.2.)

In this paper, we introduce an extension of this notion by saying that $V(f)$ has multiplicity m at p if the minimum is achieved by precisely $m+1$ choices of I. Requiring $V(f)$ to have multiplicity at least m at p is a codimension m condition on the coefficients a_I. We wish to study when points with assigned multiplicity uniquely determine a hypersurface.

3.1.2 Geometric/combinatoric

In our first approach to this problem, we start with a hypersurface with some fixed points and ask whether the hypersurface can be deformed while still containing the points.

To illustrate, consider the curves in Figures 3.1 and 3.2. In each Figure, the curve with its fixed points is shown on the left. The dual complex is shown on the right. Each edge in the dual complex whose corresponding edge in the curve has a fixed point is darkened. One can see from the figures that if the darkened subcomplex is disconnected as in Figure 3.1, then simultaneously decreasing the coefficients of a component will give a deformation. On the other hand, if the darkened subcomplex is connected as in Figure 3.2, then the curve is uniquely determined. Summarizing, we have

Proposition 3.1.1. *Let X be a tropical curve with specified fixed points in the interior of its edges. Assume every 2-polytope of the dual complex contains exactly 3 lattice points. Then X is uniquely determined if and only if the corresponding subgraph of the dual complex is connected.*

Notice that this immediately implies that at least $|\Delta|-1$ points are required for such a hypersurface to be uniquely determined, agreeing well with the classical case.

We note that for the case of curves on surfaces, essentially equivalent observations have been made in, e.g., [Mar06], Definition 4.46.

In this paper, Proposition 3.3.3 will give a condition that works for a hypersurface of any dimension and any configuration of points (allowing them to lie in higher codimensional polyhedra), and taking into account the multiplicities of Section 3.1.1. This condition lacks the appeal of the connectedness condition of Proposition 3.1.1, but we recover a similar (but much more general) statement in Theorem 3.3.5 by imposing a regularity hypothesis on the dual complex and a hypothesis on the multiplicities of the points compared to the codimension of the polyhedron they lie in.

3.1.3 Tropical Cramer's rule

In [RGST05], the authors give an algebraic way to determine whether the points uniquely determine a tropical hypersurface, which we review here. A vector $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{T}^N$ is said to be in the *tropical kernel* of a $M \times N$ tropical matrix A if for each row i,

$$\bigoplus_j x_j \odot A_{ij} = \min_j \{x_j + A_{ij}\}$$

is achieved at least twice.

Now let A be an $N \times N$ tropical matrix. The *tropical permanent* of A is

$$\bigoplus_{\sigma \in S_n} \bigodot_i A_{i,\sigma(i)} = \min_{\sigma \in S_n} \left\{ \sum_i A_{i,\sigma(i)} \right\}. \tag{3.1}$$

We say that A (or its permanent) is *tropically singular* if the minimum in the tropical permanent is achieved at least twice. A is *tropically nonsingular* otherwise.

The following are the two fundamental results.

Theorem 3.1.2 ([RGST05], Lemma 5.1)**.** *An $N \times N$ tropical matrix has a vector in its tropical kernel if and only if it is tropically singular.*

A maximal minor of a $(N-1) \times N$ tropical matrix is the tropical permanent of a $(N-1) \times (N-1)$ submatrix obtained by deleting a column.

Theorem 3.1.3 ([RGST05], Theorem 5.3)**.** *Let A be an $(N-1) \times N$ tropical matrix. Then the vector of maximal minors of A is in its tropical kernel. This vector is the unique vector in the kernel (up to tropical scaling) if and only if every maximal minor is tropically nonsingular.*

In Section 3.4, we introduce *tropical weighted matrices* which take into account the multiplicities defined in Section 3.1.1. We will state and prove generalizations of the two theorems above for tropical weighted matrices (Theorems 3.4.3 and 3.4.4). Our method

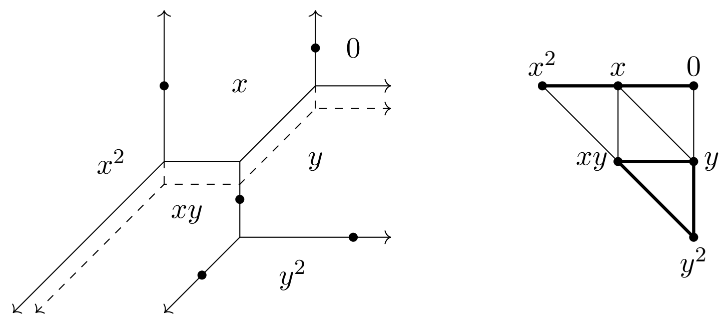


Figure 3.1. A disconnected marked subcomplex gives a deformation.

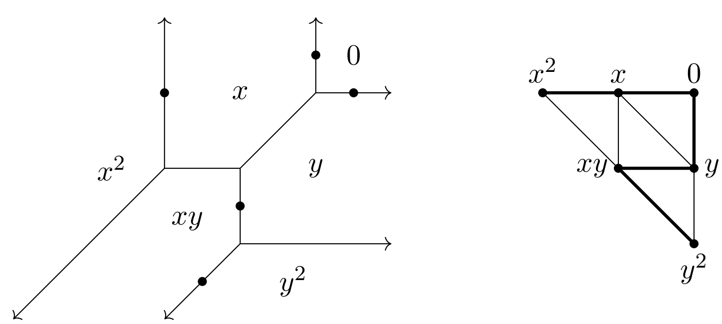


Figure 3.2. A connected marked subcomplex.

also gives a new, purely combinatorial proof of Theorem 3.1.2 (which in [RGST05] uses a lift to Puiseux series). To prove our generalization of Theorem 3.1.3 we borrow techniques and terminology from [RGST05] and [SZ93], generalizing and specializing to our situation. The restriction of our proof to the case of Theorem 3.1.3 gives a self-contained proof that fills some possibly omitted details in [RGST05] (see Remark 3.4.16).

This work grew out of discussions with Aaron Bertram, Tyler Jarvis, Lance Miller, and Dylan Zwick. The author also thanks Bernd Sturmfels for email correspondence. The author was supported by NSF Research Training Grant DMS-1246989 during parts of the work on this paper.

3.2 Tropical preliminaries

We first quickly review some basic notions from polyhedral geometry. Recall that a *polyhedron* in $\mathbb{R}^n$ is the solution to a set of linear inequalities and equations. A *face* of a polyhedron is a subset of the polyhedron obtained by changing some of the inequalites into equations. A *vertex* of a polyhedron is a zero-dimensional face and an *edge* is a one-dimensional face. A *polytope* is a compact polyhedron. A *lattice polytope* is a polytope with all of its vertices at integer points. We call a d dimensional polyhedron a d-polyhedron. A *polyhedral complex* in $\mathbb{R}^n$ is a collection of polyhedra $\mathcal{P}$ satisfying

- For every face σ of a polyhedron $P \in \mathcal{P}$, we have $\sigma \in \mathcal{P}$.
- For $P, P' \in \mathcal{P}$, we have that $P \cap P'$ is a face of both.

The *support* of a polyhedral complex is the union of the polyhedra. A *polyhedral decomposition* of a polyhedron P is a polyhedral complex whose support is equal to P.

In tropical geometry we work over the min-plus semiring $\mathbb{T}$. This ring as a set is the same as $\mathbb{R}$, but with new operations $\oplus$, $\odot$ defined by

$$a \odot b = a + b$$
$$a \oplus b = \min(a, b).$$

Fix a lattice polytope Δ. A polynomial $f \in \mathbb{T}[x_1, \ldots, x_n]$ is a formal sum

$$f = \bigoplus_{I=(i_1,\ldots,i_n)\in\mathbb{Z}(\Delta)} a_I \odot x_0^{\odot i_1} \odot \cdots \odot x_n^{\odot i_n}$$

Such a polynomial defines a polyhedral decomposition $\mathcal{E}_f$ of $\mathbb{T}^n$: for any $\mathcal{B} \subset \mathbb{Z}(\Delta)$, we get a polyhedron $P_\mathcal{B}$ (possibly empty) of this decomposition containing all points that make the monomials in $\mathcal{B}$ minimal, that is

$$P_\mathcal{B} = \left\{ \mathbf{y} \in \mathbb{T}^n : a_J + \sum_{k=1}^{n} j_k y_k = \min_{I \in \mathbb{Z}(\Delta)} \left\{ a_I + \sum_{k=1}^{n} i_k y_k \right\} \text{ for all } J \in \mathcal{B} \right\}.$$

The tropical hypersurface $V(f)$ is the subcomplex of $\mathcal{E}_f$ with polyhedra $P_\mathcal{B}$ with $|\mathcal{B}| \geq 2$. Its support is the set of $\mathbf{y}$ such that

$$\min_I \left\{ a_I + \sum_{k=0}^{n} i_k y_k \right\} \tag{3.2}$$

is achieved by more than one choice of I. By a slight abuse, we often call this support the tropical hypersurface. When we need to mention the polytope Δ, we will call $V(f)$ a Δ-hypersurface. See Figure 3.3 for a simple example.

One can construct a polyhedral decomposition $\mathcal{P}_f$ of Δ called the *dual complex*. For each subset of lattice points $\mathcal{B} \subset \mathbb{Z}(\Delta)$, the convex hull of $\mathcal{B}$ is in $\mathcal{P}_f$ if $P_\mathcal{B}$ is nonempty.

Figure 3.2 shows the dual complex to a tropical plane curve. In our figures of the dual complex, we prefer to invert the axes. This way, one can often superimpose the tropical curve on the dual complex so that every vertex lies in its corresponding 2-polyhedron, and every edge is orthogonal to its corresponding edge.

We remark that there is another (equivalent) description of the dual complex using the projection of the lower faces of the convex hull of the points $(i_1, \ldots, i_n, a_I) \in \mathbb{R}^{n+1}$. See [RGST05] for details.

In this paper, we assume that all tropical polynomials are *saturated*, that is, every monomial of f achieves the minimum in (3.2) at some point. Geometrically, this is no restriction, since for a nonsaturated polynomial, one can form its *saturation* by decreasing the offending monomials just until the new polynomial is saturated. The saturation of a polynomial still gives the same hypersurface as the original.

Definition 3.2.1. We say a dual complex $\mathcal{P}_f$ is *lattice simplicial* if for every d, the d-polyhedra of the complex contain exactly $d+1$ lattice points. For this paper, we will call a hypersurface *c-smooth* if its dual complex is lattice simplicial. Notice that lattice simplicial implies simplicial, but in dimension greater than 2, it is weaker than unimodular. We will not need those notions in this paper.

3.3 Points on a tropical hypersurface

We wish to generalize Proposition 3.1.1 by allowing the fixed points to lie in higher codimension polyhedra and allowing these higher incidence conditions. We also want to be able to consider non-c-smooth hypersurfaces. Instead of looking at a subgraph, we now consider a weighted subcomplex.

Definition 3.3.1. Let Δ be a lattice polytope, with $\mathcal{P}$ a lattice polyhedral decomposition.

Let $|P|$ be the number of lattice points contained in a lattice polytope P.

A *weighting* μ of $\mathcal{P}$ is a map

$$\mu : \mathcal{P} \to \mathbb{Z}$$

satisfying

$$0 \leq \mu(P) \leq |P| - 1$$

for all polytopes $P \in \mathcal{P}$.

Define

$$|\mu| = \sum_{P \in \mathcal{P}} \mu(P).$$

Let $\mathcal{P}^0$ be the vertices of $\mathcal{P}$. In the lattice simplicial case, of course, all the lattice points $\mathbb{Z}(\Delta)$ are vertices of $\mathcal{P}$, but in general this is not the case.

Now we need to define the analog of the connected component of the subgraph in Figure 3.1 that provided a deformation.

Definition 3.3.2. Let $L \subset \mathbb{Z}(\Delta)$ be a nonempty subset of the lattice points of Δ such that L does not contain $\mathcal{P}^0$. We say that L is *deformable* (with respect to μ) if for all $P \in \mathcal{P}$ we have either

$$|P \cap L| = 0$$

or

$$|P \cap L| \geq \mu(P) + 1.$$

We say μ is *rigid* if no such L is deformable.

Proposition 3.3.3. *Suppose $V(f)$ is a Δ-hypersurface with dual complex $\mathcal{P}$. Let $\mathbf{x}_1, \ldots, \mathbf{x}_K$ be points on $V(f)$ and $m_1, \ldots, m_K$ be positive integers such that $V(f)$ has multiplicity at* *least m_i at $\mathbf{x}_i$. Assume that no two points lie in the interior of the same polyhedron, and let μ be the weighting of $\mathcal{P}$ defined by setting $\mu(P) = m_i$ if $\mathbf{x}_i$ is in the interior of P, and $\mu(P) = 0$ otherwise.*

Then the following are equivalent:

- *$V(f)$ is the unique Δ-hypersurface passing through the $\mathbf{x}_i$ with multiplicity m_i.*
- *μ is rigid.*

Proof. First we claim that decreasing the coefficients corresponding to a subset $L \subset \mathbb{Z}(\Delta)$ by a small amount produces a tropical hypersurface that still satisfies the conditions imposed by the $\mathbf{x}_i$ and m_i if and only if L is deformable. Indeed, any point $\mathbf{x}_k$ corresponds to some $P \in \mathcal{P}$ whose lattice points correspond to the monomials minimized at $\mathbf{x}_k$. We see then that $\mathbf{x}_k$ remains on the hypersurface with multiplicity m_k if and only if there are either 0 or at least $m_k + 1 = \mu(P) + 1$ monomials of P being decreased, so the claim is proved.

If μ is not rigid, there is a deformable L not containing $\mathcal{P}^0$. Then the deformation corresponding to L is an actual deformation. (Notice that $\mathcal{P}^0$ corresponds to monomials which are uniquely minimizing on some top-dimensional polyhedron of $\mathcal{E}_f$. If all of these coefficients were decreased, the saturation of the new polynomial would be equal to a rescaling of the original f.) Hence $V(f)$ is not uniquely determined.

Now suppose g is another polynomial such that $V(g)$ satisfies the conditions imposed by the $\mathbf{x}_i$ and is distinct from $V(f)$ posed by the points. Then, for any $t \in \mathbb{T}$, $f \oplus (t \odot g)$ also satisfies the conditions imposed by the points. For $t \gg 0$, $f \oplus (t \odot g) = f$, while for $t \ll 0$, $f \oplus (t \odot g) = t \odot g$. Hence, for some value of t, decreasing t gives a deformation of the type above, which by the first claim in this proof gives a deformable L. □

In specific examples, checking that μ is rigid may not be as easy as checking whether a graph is connected, as in the case of c-smooth plane curves (see, for example Figure 3.4). However, if the hypersurface is c-smooth and the number of points is minimal, Theorem 3.3.5 will tell us that the situation is almost as nice as for plane curves. We first introduce some language.

Definition 3.3.4. We say that $P \in \mathcal{P}$ is *full* if $\mu(P)$ is as large as allowed, that is $\mu(P) = |P|-1$. We say P is *deficient* if $0 < \mu(P) < |P|-1$.

We say μ is *full* if for every P, either $\mu(P) = |P|-1$ or $\mu(P) = 0$, or equivalently, no P is deficient.

For any μ, define

$$\mathrm{Supp}(\mu) = \bigcup_{P \text{ full}} P$$

Notice that every vertex of $\mathcal{P}$ is full and thus in $\mathrm{Supp}(\mu)$.

Theorem 3.3.5. *Suppose that $\mathcal{P}$ is a lattice simplicial decomposition of Δ with weighting μ (see Definition 3.3.1) and $|\mu| = |\Delta|-1$. Then the following are equivalent:*

- *μ is rigid (see Definition 3.3.2).*
- $\mathrm{Supp}(\mu)$ *is connected.*
- $\mathrm{Supp}(\mu)$ *is connected and μ is full.*

See Figure 3.5 for an example of a curve satisfying the conditions of the theorem.

For the proof, we will need the following.

Definition 3.3.6. Let

$$\hat{\mu}(L) = \sum_{P:|P\cap L|\geq \mu(P)+1} \mu(P).$$

Clearly $\hat{\mu}(L) \leq |\mu|$. We think of $\hat{\mu}(L)$ as measuring how much of $|\mu|$ has been already "taken care of" by L. The motivation for this definition is the following lemma.

Lemma 3.3.7. *Assume $\mathcal{P}$ is lattice simplicial and $|\mu| = |\Delta|-1$. Suppose there exists L such that $\hat{\mu}(L) \geq |L|$. Then μ is not rigid.*

Proof. If $L_0 := L$ is deformable, we are done. Otherwise, there is some polytope P_0 with $1 \leq |P_0 \cap L_0| \leq \mu(P_0)$. Add in some vertices to form L_1 so that $|P_0 \cap L_1| > \mu(P_0)$. Notice that $\hat{\mu}(L_1) - \hat{\mu}(L_0) \geq \mu(P_0)$ and $|L_1|-|L_0| \leq \mu(P_0)$. It follows from these two inequalities that $\hat{\mu}(L_1) \geq |L_1|$. Continue this process. At each step we have $\hat{\mu}(L_k) \geq |L_k|$. Since $\hat{\mu}$ is bounded above, this process must terminate, that is, eventually one obtains a deformable L_k. As $|L_k| \leq \hat{\mu}(L_k) \leq |\mu| = |\Delta|-1$, this violates rigidity. $\square$

We next isolate a computation to be used in the proof.

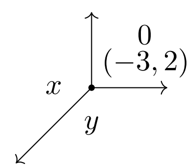


Figure 3.3. The tropical line $3 \odot x \oplus -2 \odot y \oplus 0 \odot z$

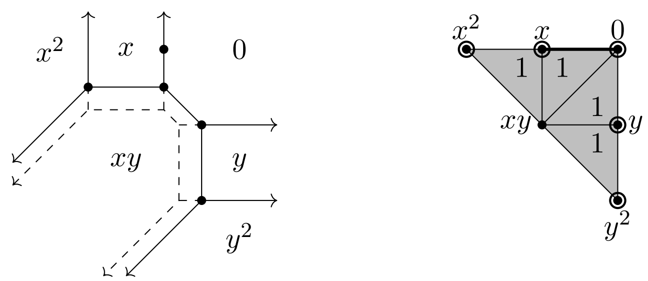


Figure 3.4. The marked dual graph looks connected, but is not full or rigid. The circled vertices give an L that is not obstructed.

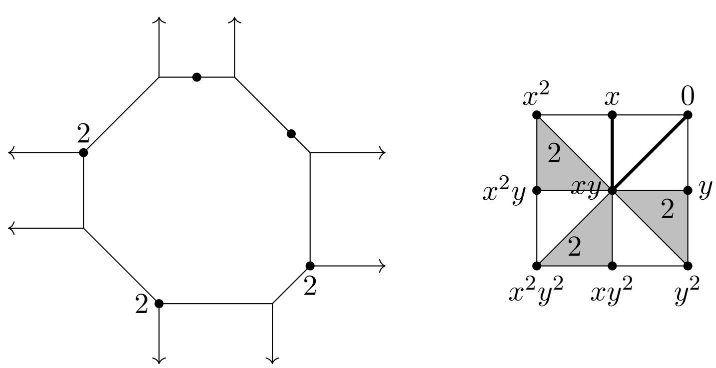


Figure 3.5. A "smooth (2,2) curve in $\mathbb{P}^1 \times \mathbb{P}^1$" with a rigid weighting with multiplicity.

Lemma 3.3.8. *For a simplicial complex $\mathcal{K}$*

$$\sum_{P \text{ maximal}} \dim P \geq |\mathcal{K}^0| - h^0(\mathcal{K}).$$

Here maximal simplexes are those which are not contained in a simplex of higher dimension, $|\mathcal{K}^0|$ is the number of vertices of $\mathcal{K}$, and $h^0(\mathcal{K})$ is the number of connected components.

Proof. Clearly it is sufficient to check this for a connected simplicial complex. It is trivially true for a complex consisting of a single point. If one attaches a d cell to a complex that shares k points, then $|\mathcal{K}_0|$ increases by $d+1-k$. On the other hand, the right hand side increases by at least $d-(k+1)$, since at worst a $(k+1)$-cell is no longer maximal. □

Corollary 3.3.9. *Suppose L is the set of lattice points of some union $\mathcal{K}$ of s components of* Supp(μ) *and suppose that there is a deficient P' with $|P' \cap L| > \mu(P')$. Then*

$$\hat{\mu}(L) \geq |L| - s + \mu(P').$$

Proof. Note that

$$\hat{\mu}(L) \geq \mu(P') + \sum_{P \subset \mathcal{K}, \text{ maximal}} \mu(P).$$

Notice that for P in the sum $\mu(P) = \dim P$ and then apply the Lemma. □

Proof of Theorem 3.3.5. Suppose Supp(μ) is connected. We will show μ is rigid. Let L be given. Pick some $v_1 \in L$ and $v_2 \notin L$. There must be a sequence $P_0, \ldots, P_k$ with P_i full and $P_i \cap P_{i+1} \neq \emptyset$ so that $v_1 \in P_0$ and $v_2 \in P_k$. Hence there must be some full P_i containing some vertex of L and some vertex not in L, and this will verify that L is not deformable.

Next, we show that if μ is not full, then μ is not rigid. Suppose first that there is some deficient polytope P and some component C_1 of Supp(μ) such that $|P \cap C_1| \geq 2$. Pick additional components C_i, $i = 2, \ldots, s$ intersecting P until $\bar{C} := \cup_{i=1}^{s} C_i$ has $|P \cap \bar{C}| > \mu(P)$. Note that $s \leq \mu(P)$. Take $L = \mathbb{Z}(\bar{C})$. Using Corollary 3.3.9, we get $\hat{\mu}(L) \geq |L| - s + \mu(P) \geq |L|$. So by Lemma 3.3.7 μ cannot be rigid.

Hence we may suppose that every deficient polytope P meets every component of Supp(μ) in at most one lattice point. Pick any component C of Supp(μ), and let $L = \{v : v \notin C\}$. This will be deformable: every full polytope will either be in C and have none of its lattice points in L or not be in C and have all of its lattice points in L, while every deficient polytope P will have at most one of its lattice points not in L (it can have at most one lattice point in C by assumption). Being deficient says that P has $\mu(P) < |P|-1$, so $|P \cap L| \geq |P|-1 > \mu(P)$ as desired. We have shown that not full implies not rigid.

It remains to show that when μ is rigid (and hence also full), $\mathrm{Supp}(\mu)$ is connected. But if $\mathrm{Supp}(\mu)$ is disconnected, one sees that the lattice points of any component will form a suitable deformable L . □

Remark 3.3.10. Both the lattice simplicial hypothesis and the hypothesis on $|\mu|$ are necessary in Theorem 3.3.5, as shown in Figures 3.6 and 3.7, respectively.

We have observed that points in $\mathbb{T}^n$ need not in general impose independent conditions on Δ-hypersurfaces. However, a single point with multiplicity $|\Delta|-1$ always determines a unique hypersurface (see Figure 3.8 for an example). In fact, somewhat amusingly, putting points on top of each other always imposes independent conditions. More precisely:

Theorem 3.3.11. *The set of coefficients of Δ-hypersurfaces passing through a point p with multiplicity m is the support of a pure codimension m polyhedral complex in $\mathbb{T}^{|\Delta|}$.*

Proof. The complex has a one-dimensional lineality space obtained by simultaneously scaling all the coefficients. Given any hypersurface meeting the point with multiplicity m, there are at least $m+1$ monomials which are minimal at p. Pick any $m+1$ of these, and now the coefficients of any other monomials can be increased. □

We conclude our study of the geometric/combinatorial method of studying genericity of points on tropical hypersurfaces.

3.4 Tropical linear algebra with multiplicities

Our goal in this section to state and prove generalizations of Theorems 3.1.2 and 3.1.3 that take into account our notion of multiplicity.

3.4.1 Definitions and statements

We begin by introducing the definition of a *tropical-weighted* (henceforth *tw-*) *matrix.*

Definition 3.4.1. A $K \times N$ tw-matrix A is a $M \times N$ matrix, together with a partition $\sum_{i=1}^{M} m_i = K$, $m_i \geq 1$.

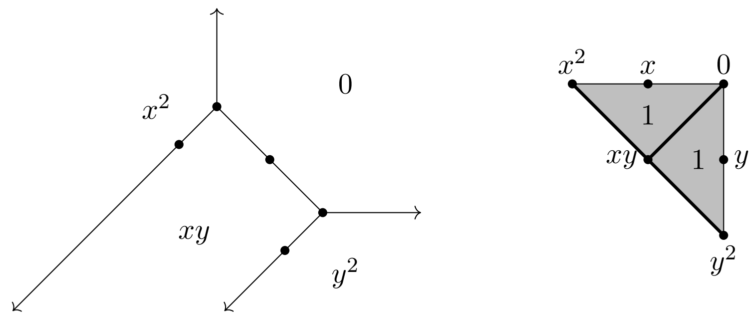


Figure 3.6. A singular conic with a rigid weighting.

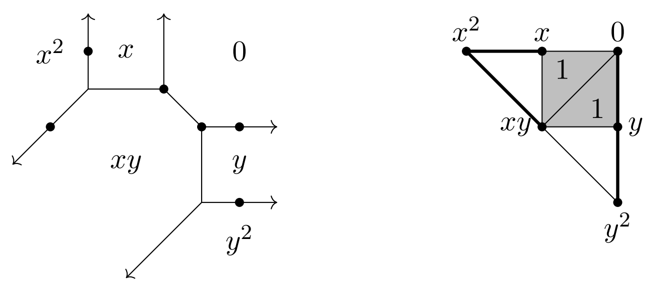


Figure 3.7. This weighting is not full but is rigid, showing that the hypothesis $|\mu|= |\Delta|-1$ is necessary in Theorem 3.3.5.

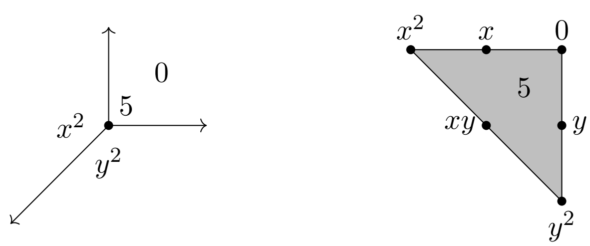


Figure 3.8. An extreme example of a weighting with higher multiplicity.

For a $K \times N$ tw-matrix, we say that a vector $\mathbf{x}$ is in the tw-kernel of A if for each i, $\min_j\{x_j + A_{ij}\}$ is achieved at least $m_i + 1$ times.

The tw-permanent of an $N \times N$ tw-matrix A is

$$\min_{\mathcal{I}} \left\{ \sum_i \sum_{j \in I_i} A_{ij} \right\},$$

where the minimum is over all partitions $\mathcal{I} = \{I_i\}_{i=1}^K$ of $\{1, \ldots, N\}$ with $|I_i| = m_i$. The I_j themselves are ordered, but the elements of I_j are not. We say A (or its tw-permanent) is tw-singular if the minimum above is obtained more than once.

Remark 3.4.2. One could also describe the tw-permanent by saying that it is the minimum sum obtained by choosing, for each i, m_i entries from row i so that exactly one entry is chosen from each column. We also note that the value of the tw-permanent is the same value that would be obtained by repeating the ith row m_i times and taking the usual tropical permanent (3.1) (but such a permanent would automatically be singular for a nontrivial partition).

When A is $(N-1)\times N$, a maximal minor of A is the tw-permanent of the $(N-1)\times(N-1)$ tw-matrix formed by deleting a column of A.

Our main theorems of this section are:

Theorem 3.4.3. *An $N \times N$ tw-matrix is tw-singular if and only if there is a vector in its tw-kernel.*

Theorem 3.4.4 (Tropical Weighted Cramer's Rule)**.** *Let A be an $(N-1) \times N$ tw-matrix. Then the vector of maximal minors is in the tw-kernel of A. Furthermore, this vector is unique up to scaling if and only if every maximal minor of A is tw-nonsingular.*

These will be proved in Sections 3.4.3 and 3.4.4. Notice that in the case that $m_i = 1$ for all i, these theorems recover Theorems 3.1.2 and 3.1.3.

As an example, consider the 3×4 tw-matrix:

$$3 \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

Here the 3 on the left indicates that $m_1 = 3$. The vector of maximal minors is $\begin{bmatrix} 9 & 8 & 7 & 6 \end{bmatrix}$, which is in the tw-kernel, as the values of $x_j + A_{ij}$ are $\begin{bmatrix} 10 & 10 & 10 & 10 \end{bmatrix}$. One easily sees that this is unique (up to tropical scaling), and in fact, a 3×3 tw-matrix with $m_1 = 3$ is always tw-nonsingular.

As a second example, consider

$$\begin{array}{c} 2 \\ 1 \end{array} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 3 \end{bmatrix}$$

The vector of maximal minors is $\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}$. However, the vector $\begin{bmatrix} 1 & 1 & 1 & 99 \end{bmatrix}$ is also in the tw-kernel. And indeed, the 4th minor is tw-singular, with $\{1,2\},\{3\}$ and $\{1,3\},\{2\}$ being permutations achieving the minimum value of 1 and having $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ in its tw-kernel.

We can summarize the geometric consequences as follows.

Corollary 3.4.5. *Let Δ be a n-dimensional lattice polytope and set $N = |\Delta|$. Let $\{\mathbf{x}_i\}$ be a collection of K points in $\mathbb{T}^n$, and m_i multiplicities so that $\sum m_i = N - 1$. Let A be the $(N-1) \times N$ tw-matrix formed by evaluating lattice points of Δ at $\mathbf{x}_i$. Then there is a tropical hypersurface passing through each $\mathbf{x}_i$ with multilplicity m_i. That hypersurface is unique if and only if the maximal minors of A are tw-nonsingular.*

3.4.2 Hypergraphs

Graphs provide a convenient way to keep track of patterns formed when solving optimization problems using tw-matrices. Because we are allowing multiplicities, we will need to work with hypergraphs.

Definition 3.4.6. A *hypergraph* G is a set V of vertices, together with a set E of nonempty subsets of V called edges. We say an edge e *touches* a vertex w if $w \in e$. The *valency* of a vertex is the number of edges touching it.

A *path* in a hypergraph is a sequence $v_1, e_1, v_2, e_2, \ldots, v_n$, with $v_i \in V$ and $e_i \in E$ such that v_i and v_{i+1} are distinct and both in e_i. There is an equivalence relation on V where $v \sim w$ if there is a path with $v = v_1$ and $w = v_n$. An equivalence class of V, together with all edges touching any vertex in the class, is called a *connected component* of G. The hypergraph is *connected* if there is only one connected component. We call a path a *cycle* if $v_1 = v_n$ and no edge is repeated. A cycle is *simple* if no vertex is repeated besides $v_1 = v_n$. Notice that if a graph has a cycle, it must also have a simple cycle.

The *edge total* $e(G)$ of a hypergraph is $\sum_{e \in E}(|e|-1)$. Notice that if G is a graph then the edge total is equal to the number of edges. The vertex total is $v(G) = |V|$.

Our motivation comes from the following definitions.

Definition 3.4.7. When considering a tw-matrix A as in Definition 3.4.1, we say a hypergraph is a *linkage hypergraph* if it has N vertices (corresponding to columns of A) and M edges (corresponding to rows of A) such that edge i touches exactly $m_i + 1$ vertices.

Now let A be a non-negative $K \times N$ tw-matrix. A linkage hypergraph complementary to A is a linkage hypergraph such that $A_{ij} = 0$ whenever i touches vertex j.

Notice that a non-negative A has a linkage hypergraph complementary to it if and only if $\mathbf{0}$ is in its tw-kernel.

Now let Y be a non-negative $K \times N$ tw-matrix. The support hypergraph of Y is the hypergraph formed by having edge i touch vertex j whenever $Y_{ij} > 0$.

Lemma 3.4.8. *The support hypergraph of Y contains a simple cycle if and only if Y is not uniquely determined by the data of its support, row sums, and column sums.*

Proof. Let $i_1, j_1, i_2, j_2, i_3, \ldots, j_\ell, i_{\ell+1} = i_1$ be a simple cycle of the support hypergraph of Y. One can add a small amount to entries (i_k, j_k) and subtract the same amount from entries (i_{k+1}, j_k). This new matrix still has the same row and column sums and the same support.

Now suppose $Y' \neq Y$ has the same support and row and column sums as Y. By taking convex combinations of Y and Y', we see that there is a deformation of the non-zero entries of Y that preserves the row and column sums. Suppose Y_{i_1,j_1} decreases in this deformation. Thus, there must be a j_2 with Y_{i_1,j_2} increasing (in order to preserve the i_1 row sum). Then, there must be an i_2 with Y_{i_2,j_2} decreasing (to preserve the j_2 column sum). Continue in this way until some "i" is repeated, say $i_m = i_s$, with $m < s$. Then $j_{m+1}, i_{m+1}, j_{m+2}, \ldots, i_s = i_m, j_{m+1}$ is a cycle in the support hypergraph. $\square$

Notation 3.4.9. The following construction allows us to use results about graphs to help prove results about hypergraphs. Given a hypergraph G, we may construct a (not hyper) graph $\hat{G}$ by replacing each edge $e \in E_G$ with a tree connecting all vertices touched by e. $E_{\hat{G}}$ comes equipped with a surjection ψ onto E_G, where an edge of $\hat{G}$ is associated to the edge of G that gave rise to it. Notice that $e(G) = e(\hat{G})$ and $\hat{G}$ is connected if and only if G is.

Proposition 3.4.10. *A hypergraph G has a simple cycle if and only if*

$$e(G) + \text{ \# of components } \geq v(G) + 1.$$

Proof. The result is standard for (nonhyper) graphs. If G has a simple cycle, then so does $\hat{G}$, so the inequality is satisfied.

Now, assume we have the inequality. Then $\hat{G}$ has a simple cycle $v_1, e_1, v_2, e_2, \ldots, v_n$. This gives a (possibly not simple) cycle $v_1, f_1, v_2, f_2, \ldots, v_n$ in G, where $f_i = \psi(e_i)$. Since ψ only collapses trees, there must be at least two f_i distinct, so by changing the base point of the loop we may assume $f_1 \neq f_{n-1}$. Now, if this cycle in G is not simple, say $f_j = f_k$, we may replace it by $v_1, f_1, \ldots, v_j, f_j, v_{k+1}, \ldots v_n$. Repeating this process, we obtain a simple cycle in G. $\square$

Definition 3.4.11. For this paper, we define a *good orientation* of a (nonhyper) graph to be an orientation such that every vertex has exactly one outgoing edge. It is easy to see that this is equivalent to giving a bijection $out : V \to E$ such that $out(v)$ is an edge touching v. This is a special case of a good orientation of a hypergraph, which is a surjection $out : V \to E$ such that $out(v)$ touches v and $|out^{-1}(e)|= |e|-1$.

Notice that a good orientation for $\hat{G}$ gives one for G by composing with ψ.

Proposition 3.4.12. *Given a connected hypergraph G with $e(G) = v(G)$, then there are at least two good orientations for V.*

Proof. There is a unique simple cycle in $\hat{G}$. There are two distinct good orientations for this cycle — one for each way of going around the cycle. Excluding the edges in the cycle from $\hat{G}$ gives a forest, with each tree having a distinguished vertex that was part of the cycle. One can extend the good orientation to the forest by taking the flow into the distinguished vertex (each vertex v will be the source of the first edge in the unique path from v to the distinguished vertex).

It remains to check that these orientations are distinct after composing with ψ. But this could only fail if ψ collapses the cycle — which cannot happen because ψ only collapses trees. $\square$

Corollary 3.4.13. *If A is a non-negative $N \times N$ tw-matrix and has a connected complementary linkage hypergraph, then A is tw-singular.*

Proof. By the lemma the complementary linkage hypergraph has two good orientations. A good orientation of a complementary hypergraph is the same as a choice of partition $\mathcal{I}$ that certifies that tw-perm $A \leq 0$ (via $I_i = out^{-1}(i)$). But as A was assumed non-negative, we have tw-perm $A \geq 0$, so tw-perm $A = 0$ and A is tw-singular. □

3.4.3 More lemmas and the proof of Theorem 3.4.3

Lemma 3.4.14. *For any $N \times N$ tw-matrix A, one can rescale the rows and columns so that every entry is non-negative and* tw-perm $A = 0$.

Proof. First we do the case $m_i = 1$ for all i. By permuting rows and columns and rescaling, we may assume that the diagonal is the minimizing permutaion and that every entry on the diagonal is 0. The problem now is to show that the matrix can be further rescaled to eliminate the negative entries without disturbing the diagonal.

We construct a labeled graph $G(A)$ by taking complete di-graph on N labeled vertices, with the edge from i to j labeled by A_{ij}. For any directed path γ in $G(A)$, we define its path sum $p(\gamma)$ to be the sum of all the labels of the edges in the path. Notice that the path sum of any simple cycle corresponds to a (cyclic) permutation in the formula for the tropical permanent, so we see that the tropical permanent being equal to zero implies that the path sum of any simple cycle is non-negative.

Notice that if we subtract c from row i and add c to column i, we still have a matrix with each diagonal entry equal to zero and tropical permanent equal to zero. Hence it is enough to pick c_i so that $A_{ij} - c_i + c_j \geq 0$.

Pick

$$c_i = \min \{p(\gamma) : \gamma \text{ is a simple path starting at } i\}.$$

Here a simple path is one that uses any vertex at most once.

Now we check $A_{ij} - c_i + c_j \geq 0$. There is a simple path γ from vertex j with $p(\gamma) = c_j$. If γ does not meet vertex i, then we can form a simple path γ' starting at i by concatenating the edge from i to j with γ. Then $c_i \leq p(\gamma') = c_j + A_{ij}$, as desired. If γ meets vertex i, we can split γ at vertex i into two simple paths: γ_1 from j to i and γ_2 the rest of it (possibly trivial). We have $p(\gamma_1) + p(\gamma_2) = c_j$. Now, we can form a simple cycle by concatenating γ_1 with the edge from i to j. Hence $p(\gamma_1) + A_{ij} \geq 0$. Furthermore, γ_2 is a simple path from i, so $c_i \leq p(\gamma_2)$. Hence

$$A_{ij} - c_i + c_j \geq -p(\gamma_1) - c_i + p(\gamma_1) + p(\gamma_2) \geq 0,$$

as desired.

For the general case, we can permute and rescale it so that the optimizing partition is the "diagonal" one, i.e., $I_1 = \{1, \ldots, m_1\}$, $I_2 = \{m_1 + 1, \ldots m_1 + m_2\}$, etc., and so that each of these entries are 0. Then we form a square (nonweighted) tropical matrix by repeating row i m_i times. Then run the argument above. Notice that two vertices (say i and k) corresponding to a row repeated in this way will have all edges between them labeled 0. Hence $c_i = c_k$, and the resulting matrix will have the same pattern of repeated rows. One can then reidentify these rows to get the desired tw-matrix. □

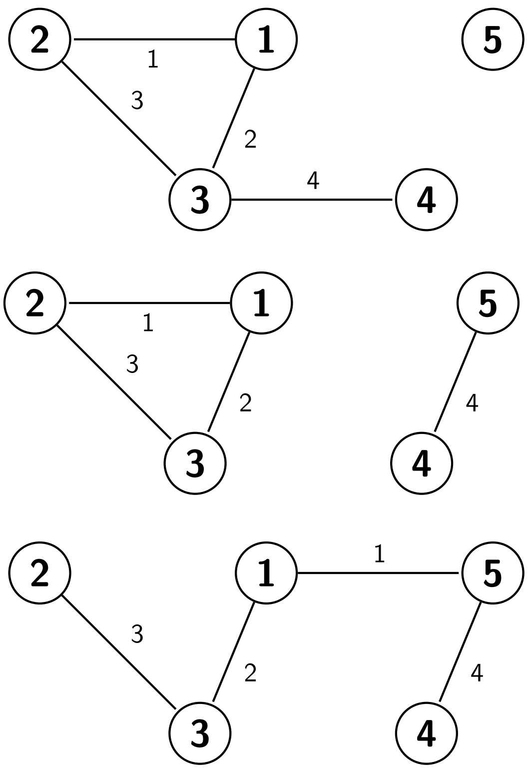
The following lemma is used in the proof of both theorems. The full strength of it is needed only for the $(N-1) \times N$ case.

Lemma 3.4.15. *Suppose A is a non-negative $K \times N$ tw-matrix with $K \geq N - 1$. Let G be a linkage hypergraph complementary to A. If G is not connected, then we may rescale rows and columns to produce A' that is non-negative and has two complementary connected linkage hypergraphs.*

Proof. See Figure 3.9 for an illustration of the proof.

Given any connected component of G with edges E and vertices V, one can perform the following operation (*): Subtract from the rows of A corresponding to E and add to the columns of A corresponding to V to form A'. Do this as much as possible without violating non-negativity, i.e., add and subtract $\min_{i \in E, j \notin V} A_{ij}$. Notice that G is also complementary to A'. Let (i^*, j^*) be the indices achieving the minimum above. Then $A'_{i^* j^*} = 0$, so we may replace any vertex touching edge i^* with vertex j^* and get a new linkage hypergraph G' that is complementary to A'.

Let $M(G)$ be the minimal number of vertices of a connected component of G that has a cycle (such a component exists by Proposition 3.4.10), and let C be one of these components. We will perform operation (*) with this component. To choose the vertex touching i^* to



$$\begin{bmatrix} 0 & 0 & 3 & 3 & 3 \\ 0 & 4 & 0 & 4 & 5 \\ 5 & 0 & 0 & 3 & 4 \\ 6 & 7 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 3 & 3 & 2 \\ 0 & 4 & 0 & 4 & 4 \\ 5 & 0 & 0 & 3 & 3 \\ 6 & 7 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 3 & 1 & 0 \\ 0 & 4 & 0 & 2 & 2 \\ 5 & 0 & 0 & 1 & 1 \\ 6 & 7 & 0 & 0 & 0 \end{bmatrix}$$

Figure 3.9. An example of running the proof of Lemma 3.4.15 on an explicit matrix. Node labels refer to columns, edge labels refer to rows. To move from the first picture to the second, we used case (2), and from the second to the third used case (1). In the last picture, one could just as well have edge 1 connect nodes 2 and 5.

be replaced by j^*, we must consider the components $\{C_\alpha\}$, $\alpha = 1, \dots, k$, formed from C by deleting edge i^*.

1. If $k \leq m_{i^*}$, then there is some C_β that contains at least two vertices of edge i^*. Pick one of these vertices and replace it with j^* to form G'. Then we see that G' has fewer components than G.

2. If $k = m_{i^*} + 1$, then we cannot reduce the number of components of G. Instead, let $\bar{C}$ be the subgraph $\bigcup_\alpha C_\alpha$ and notice that

$$\begin{aligned} e(\bar{C}) + \text{ \# of components of } \bar{C} &= e(\bar{C}) + m_{i^*} + 1 \\ &= e(C) + 1 \\ &\geq v(C) + 1 \\ &= v(\bar{C}) + 1 \end{aligned}$$

   so Proposition 3.4.10 implies that there is some C_β that has a cycle. Pick the vertex of C_β touched by i^* and replace it with j^*. Then we see that C_β will be a component of G', which will imply that $M(G') < M(G)$.

Hence at each step, either the number of connected components decreases, or $M(G)$ decreases. Since $M(G)$ is non-negative, it cannot decrease forever, so eventually the number of components must decrease. Furthermore, notice that the final step will be of type (1), and so there are two distinct choices giving hypergraphs complementary to A'. □

Remark 3.4.16. In [RGST05], something like Lemma 3.4.15 appears to be being used implicitly. On page 19 (of the arXiv version), the authors start with a $(N-1) \times N$ tropical matrix C that is assumed to have nonsingular minors. From Theorem 2.4 in [SZ93], one knows that there is an associated linkage tree (from which the optimizing permutations of the minors can be extracted). The matrix C is then rescaled so that it is non-negative and has at least two zeros on each row and one zero in each column. It is then claimed that the linkage tree of C is complementary to (the rescaled version of) C. For smaller values of N it is easy to see that if the zero patterns do not form a tree then there is a singular minor (with tropical permanent equal to 0), but for larger N this is not obvious to us. For example, the pattern of zeros could be represented by a graph as in Figure 3.10.

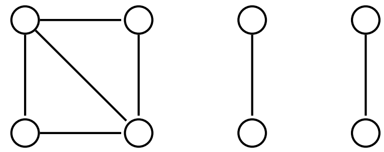


Figure 3.10. An interesting zero pattern

In this case, there is no minor that is obviously singular. No minor has permanent equal to zero and which minor is singular will depend on the matrix itself and not only on the zero pattern. It seems to us that an argument like Lemma 3.4.15 is necessary here.

Notation 3.4.17. If a matrix A has a vector $\mathbf{x}$ in its tw-kernel, one can scale column j by x_j, forming a new matrix which has $\mathbf{0}$ in the tw-kernel. Then, one can scale each row so that the minimal entry is zero. We call the matrix obtained this way $A_{\mathbf{x}}$.

Proof of Theorem 3.4.3. First, assume that A has a vector $\mathbf{x}$ is its tw-kernel. Then, there is a linkage hypergraph complementary to $A_{\mathbf{x}}$. We may assume that it is connected (after possibly rescaling) by Lemma 3.4.15. Hence by Corollary 3.4.13 A is tw-singular.

Now, suppose we are given two partitions I and J . By Lemma 3.4.14 we may assume that A has non-negative entries that every entry corresponding to either partition is 0, that is $A_{ij} = 0$ whenever $j \in I_i$ or $j \in J_i$. Now, take $K = \{k : J_k \neq I_k\}$, and let $L = \{\ell : \ell \in J_k \cup I_k \text{ for some } k \in K\}$.

One can check that $\sum_{k \in K} m_k = |L|$ (that is, the submatrix determined by L and K is "tw-square").

Now notice that $\mathbf{0}_K$ is in the tw-kernel of the submatrix determined by K and L. We wish to extend this submatrix. Take $\epsilon = \min_{i \notin K, j \in L} A_{ij}$ and then add ϵ to each row in K and subtract ϵ from each column in L. We see now that there is a zero entry in some row $i^* \notin K$ and column $j^* \in L$. We can append k^* to K and $J_{k^*} = I_{k^*}$ to L and we still have the property that $\mathbf{0}_K$ is in the tw-kernel of the submatrix determined by K and L. We may repeat this until $|L| = |K| = N$. The resulting matrix will have $\mathbf{0}$ in its tw-kernel, so the theorem is proved. $\square$

3.4.4 Stochastic and transportation polytopes and the proof of Theorem 3.4.4

Definition 3.4.18. Let D be the polytope of weighted doubly stochastic square $(N-1)\times(N-1)$ tw-matrices—that is, we require $\sum_j A_{ij} = 1$ for each i, and $\sum_i m_i A_{ij} = 1$ for each j.

Remark 3.4.19. Given a polytope defined by a collection of equality and inequality constraints, we see that a point of the polytope is a vertex if it is the unique solution to the system of equations formed by replacing all the tight inequality constraints with equalities (and keeping all the equality constraints). Since in our situation, the inequality constraints are $A_{ij} \geq 0$, we see that a vertex of D is a matrix whose entries are uniquely determined by its support and the constraints on the row and column sums. By Lemma 3.4.8, we see that this is equivalent to the support hypergraph not containing a cycle.

Definition 3.4.20. For any two $K \times N$ tw-matrices, define the weighted inner product $\langle\cdot,\cdot\rangle_w$ by

$$\langle A, Y\rangle_w = \sum_{i,j} m_i A_{ij} Y_{ij}.$$

Lemma 3.4.21. *We have:*

1. *The vertices of D are in bijection with partitions $\mathcal{I}$ as in* (3.4.1) *via $(Y_{\mathcal{I}})_{ij} = \frac{1}{m_i}$ if $j \in I_i$ and* 0 *otherwise.*

2. $$\min_{Y\in D} \langle A, Y\rangle_w = \text{tw-perm}(A)$$

   and the minimizing Y is unique if and only if A is tw-nonsingular.

Proof. To prove the second statement from the first, note that

$$\langle A, Y_{\mathcal{I}}\rangle_w = \sum_i \sum_{j\in I_i} A_{ij}.$$

Thus we see that the tw-permanent is equal to $\min_{\mathcal{I}} \langle A, Y_{\mathcal{I}}\rangle_w$. Since the minimum over D is achieved at a vertex, we see that (2) follows from (1).

To prove (1), we first we claim that $Y_{\mathcal{I}}$ is a vertex of D. Since each column has only one non-zero entry, that entry is determined the column sum, hence $Y_{\mathcal{I}}$ is determined by its support and is a vertex by Remark 3.4.19.

Now start with any Y a vertex of D. Suppose there is pair k, ℓ such that $Y_{k\ell} > 0$ and column ℓ has more than one nonzero entry, or equivalently $Y_{k\ell} < \frac{1}{m_k}$. All entries in row k are less than or equal to $\frac{1}{m_k}$, and they all sum to 1. It follows that row k must have at least two entries strictly less than $\frac{1}{m_k}$. Both columns containing these entries have more that one nonzero entry each. Translating this into the language of support hypergraphs, we have shown that if an edge contains a vertex with valence greater than 2, then it contains another vertex with valence greater than 2. It follows then that the support hypergraph contains a simple cycle, contradicting Remark 3.4.19. We conclude then that any vertex of D has exactly one nonzero entry in each column. From this it quickly follows that Y has the form of $Y_{\mathcal{I}}$.

That uniqueness of the minimizing Y is equivalent to tw-nonsingularularity is clear. □

Definition 3.4.22. The weighted transportation polytope T is the set of non-negative $(N-1) \times N$ tw-matrices with row sums equal to N and weighted column sums equal to $N-1$, that is $\sum_j A_{ij} = N$ for each i, and $\sum_i m_i A_{ij} = N-1$ for each j.

There are N embeddings $\phi_1, \ldots, \phi_N$ of the space D of $(N-1) \times (N-1)$ tw-matrices into the space of $(N-1) \times N$ tw-matrices given by inserting a column of zeros. Notice that the Minkowski sum $\sum_{j=1}^N \phi_j(D)$ is contained in T.

Notice that Remark 3.4.19 applies also to T.

Lemma 3.4.23. *There is a bijection between vertices of the transportation polytope T and connected linkage hypergraphs G, given by taking the support hypergraph of a vertex. Every vertex of T is in the Minkowski sum $\sum_{j=1}^N \phi_j(D)$.*

Proof. First, suppose we are given a connected support hypergraph G. Form $\hat{G}$ as in Remark 3.4.9. Notice that $\hat{G}$ is a tree, so for each vertex j we can consider the function $out_j : V(\hat{G}) - \{j\} \to E(\hat{G})$ that assigns to each vertex $\ell \neq j$ the first edge in the unique path from ℓ to j. Let E_{xy} be the matrix with zeros everywhere except for a 1 in the (x, y) entry. Let $k(\ell) = \psi \circ out_j(\ell)$ (here ψ is as in Remark 3.4.9) and define

$$D_j = \sum_{\ell \neq j} \frac{1}{m_{k(\ell)}} E_{k(\ell),\ell}.$$

It is easy to check $D_j \in \phi_j(D)$, so $Y = \sum_j D_j$ is contained in the Minkowski sum. We also see that Y has support hypergraph G and $Y \in T$. By Proposition 3.4.10, G has no cycle, so by Remark 3.4.19 it is a vertex.

Now, suppose you have a vertex Y of T. We must check that the support hypergraph is a connected linkage hypergraph. First we claim that no row i can have less than $m_i + 1$ nonzero entries. Otherwise, since the row sums are N, some entry must be greater than or equal to N/m_i. But then the weighted column sum of the column containing that entry is at least N, so Y could not be in T.

Now, if any row has too many nonzero entries, then the support hypergraph must by Proposition 3.4.10 contain a simple cycle and Y would not be a vertex (see again Remark 3.4.19). So we conclude that the support hypergraph of Y is a linkage hypergraph.

If the support hypergraph of Y is not connected, then again by Proposition 3.4.10 it contains a simple cycle. We conclude that the support hypergraph of Y is a connected linkage hypergraph. □

Corollary 3.4.24. *T is precisely the Minkowski sum $\sum_{j=1}^{N} \phi_j(D)$.*

Proof. We have noted that the Minkowski sum is contained in T. For the other inclusion, since both sets are convex polytopes, it is sufficient to show that any vertex of T is in the Minkowski sum, which is the second statement in Lemma 3.4.23. □

Remark 3.4.25. The (unweighted) transportation polytope can be given the following physical interpretation. Suppose one has N factories, each of which produce $N-1$ units of a product. Suppose there are $N-1$ cities, each of which consume N units of the product. The cost to transport a unit of the product from a factory to a city is given in the matrix A. Then T is the feasible region for this optimization problem and $Y \mapsto \langle A, Y \rangle$ is the cost function.

Lemma 3.4.26. *Consider the problem*

$$\min_{Y \in T} \langle A, Y \rangle_w . \tag{3.3}$$

The solution to (3.3) *is unique if and only if A has tw-nonsingular minors.*

Proof. By Corollary 3.4.24, we see that (3.3) can be rewritten as

$$\min_{\{Y_j \in \phi_j(D)\}_j} \left\langle A, \sum_j Y_j \right\rangle_w \tag{3.4}$$

$$= \sum_j \min_{Y_j \in \phi_j(D)} \langle A, Y_j \rangle_w \tag{3.5}$$

But by Lemma 3.4.21, we know that the summands in (3.5) have solutions equal to the minors of A which are unique if and only if the minors are tw-nonsingular. □

Proof of Theorem 3.4.4. First we check that the vector of maximal minors is in the tw-kernel. By rescaling the columns, we may assume the the first row has every entry equal to 0. The *lower* minors of A are the minors obtained by deleting the first row and $m_1 + 1$ columns.

Notice that the value of the ith maximal minor is equal to the value of the smallest lower minor contained in it.

Let L be the set of $N - 1 - m_1$ indexes defining the minimal lower minor. Hence for any $i \notin L$, the ith maximal minor has this value, and is minimal among maximal minors. Hence we see that the vector of maximal minors is in the kernel of the first row. As the Theorem is invariant under permutation of rows, we are done.

Next, suppose A has a singular minor. By Theorem 3.4.3, the singular minor has a vector in its tw-kernel. Extending this vector by inserting any sufficiently large entry will create a vector in the tw-kernel of A.

Now suppose A has two elements $\mathbf{x}$ and $\mathbf{y}$ of its kernel. There is a linkage hypergraph complementary to $A_{\mathbf{x}}$. If it is not connected, then by Lemma 3.4.15 we may rescale to produce a non-negative A' with two complementary connected linkage hypergraphs. Each of these hypergraphs corresponds to a vertex of T, and since they are complementary to A' they achieve the optimal value of 0 in (3.3), so by Lemma 3.4.26, A' and hence also A has a tw-singular minor. The same argument applies to $A_{\mathbf{y}}$.

So now we may assume that both $A_{\mathbf{x}}$ and $A_{\mathbf{y}}$ have connected complementary linkage hypergraphs. Again, each of these trees corresponds to an optimal solution to (3.3) by Lemma 3.4.23, so if they are distinct, we must have a singular minor.

So finally, we may assume that $A_{\mathbf{x}}$ and $A_{\mathbf{y}}$ both have the same connected complementary linkage hypergraphs. Let $\min_i^x = \min_j\{A_{ij} + x_j\}$ and $\min_i^y = \min_j\{A_{ij} + y_j\}$. By construction, we see that for any edge i touching vertex j in this hypergraph, we have

$x_j + A_{ij} = \min_i^x$ and $y_j + A_{ij} = \min_i^y$, hence $x_j - y_j = \min_i^x - \min_i^y$. It follows that if there is an edge from j to k in the support hypergraph, then $x_k - y_k = x_j - y_j$. But since the hypergraph is connected, this implies that $\mathbf{x}$ is a tropical scalar multiple of $\mathbf{y}$. □

3.4.5 Spaces of hypersurfaces with negative expected dimension are empty

First we prove a lemma.

Lemma 3.4.27. *Suppose the Δ-hypersurface in Corollary 3.4.5 is uniquely determined. Then it satisfies the constraints exactly, that is, exactly m_i monomials are minimized at the point $\mathbf{x}_i$.*

Proof. Suppose the constraints are not satisfied exactly; that is, the matrix formed by evaluating the monomials at points can be rescaled to a matrix A which is non-negative and has at least m_i zero entries on each row, and more than m_i zero entries on at least one row. This implies that there are at least two distinct complementary linkage hypergraphs to this matrix. If either is disconnected, then we may apply Lemma 3.4.15, so in any case we can find two connected complementary linkage hypergraphs. It follows that the solution to (3.3) is not unique, so the hypersurface was not uniquely determined. □

A consequence of this lemma is a theorem that shows that no analogue of Nagata's conjecture for curves is needed (in any dimension) for our version of multiplicity. That is, for any prescribed multiplicities so that the expected dimension is negative, if the points are in general position, then there are no hypersurfaces satisfying the constraints. More precisely:

Theorem 3.4.28. *Let $S = \mathbb{T}^{|\Delta|}$. There is a polyhedral complex $Z \subset S^K$ with each cell of codimension at least one so that if K points with multiplicities m_i have $\sum m_i \geq |\Delta|$, then for choices of points not in Z, there is no tropical hypersurface satisfying the constraints.*

Proof. Pick a subset $I \subset \{1, \ldots, K\}$ and m_i' for $i \in I$ so that $m_i' \leq m_i$ and $K' := \sum_{i \in I} m_i' = |\Delta| - 1$. Then, in $S^{K'}$ let Z' be the union of hypersurfaces determined by the maximal minors of the tw-matrix determined by I and m_i'. For any point $\mathbf{x} \in S^{K'} - Z'$, let $f_{\mathbf{x}}$ be the unique tropical Δ-polynomial determined by the conditions m_i'. Then the bad locus (where there exists a hypersurface satisfying the constraints) is contained in

$$Z = Z' \times S^{K-K'} \cup \{\mathbf{x} \in S^{K'} - Z', \mathbf{y} \in S^{K-K'} : y_i \in V(f_{\mathbf{x}}) \text{ for all } i\}.$$

(Here $\mathbf{y} = (y_i)_{i=1}^{K-K'}$, $y_i \in S$.) Indeed, assume the points avoid Z. First if I is a proper subset, then avoiding the first set in the union above implies that I and m_i' determine a unique Δ-hypersurface. Avoiding the second set means that there is at least one point (for some $i \notin I$) that is not on that hypersurface.

Now suppose $I = \{1, \ldots, K\}$. Again, avoiding the first set in the union implies that there is a unique hypersurface satisfying the conditions imposed by the m_i'. But then by Lemma 3.4.27 it does not satisfy the conditions imposed by the m_i. $\square$

APPENDIX A

CODE

The following code, as well as instructions on how to use it, is available at:

https://bitbucket.org/drew_j/strange-duality-code.

A.1 therings.sage

This file contains definitions of rings used in several of the other files.

```
"""
This is the coefficient ring and power series ring for A, B, f
   , g, etc. to live in.
"""
big_poly_ring.<n,r,L,K,L2,KL,K2,chiL,chiS,deg> =
   PolynomialRing(QQ)
big_power_series_ring.<z> = PowerSeriesRing(big_poly_ring)

inf_poly_x.<x> = InfinitePolynomialRing(QQ)
```

A.2 Localization.sage

This file implements the techniques discussed in Section 2.5.

```
"""
Performs the localization computations as suggested in Remark
   5.4 of "On the Cobordism Class of the Hilbert Scheme of a
   Surface"
[EGL], using the techniques of "Botts Formula and Enumerative
   Geometry" [ES].
We can then compute the coefficients of the LHS in Theorem
   5.3, (asuming S= P^2) and thereby determine A and B.

We replace the "r" in [ES] with "n" (the number of points on
   the Hilbert Scheme).
We replace the "n" in [ES] with "m", since n is already used.

We use this to compute the A, B, and Phi series of "On the
   Cobordism Class of the Hilbert Scheme of a Surface [EGL].
```

```
The most important user functions in this file are Euler and
   EulerP2.
"""



"""
This represents the representation ring of the maximal torus
   in P^2 = P(V).
Used in all the functions below.
"""
rep_ring = LaurentPolynomialRing(QQ,"la0,la1,la2")
la = rep_ring.gens()


def Tripartitions(n):
    """
    A generator for Tripartitions of n.
    """
    for i in range(0,n+1):
        for j in range(0,n+1-i):
                k = n-i-j
                for p in CartesianProduct(Partitions(i),
                   Partitions(j), Partitions(k)):
                    yield p

def repOnkH(k,B):
    """
    This gives the rep on fiber of (kH)_n at the fixed point
       corresponding to B.
    See [ES], (4.6).
    Notice that \mathcal L in [ES] is equivalent to H_n in our
        notation.
    (Here H is the hyperplane class.)
    (In [ES] (-)_n has a different meaning.)
    """
    return prod(( la[i]**(k*sum(B[i])) for i in range(3) ))

def DmB(B,m):
    """
    This is the D_n(B) in [ES], near the bottom of page 17.
    """
    return [ (m-r-s, r, s) for r,s in B[0].cells() ] + [ (s, m
       -r-s, r) for r,s in B[1].cells() ] + [ (r, s, m-r-s)
       for r,s in B[2].cells() ]

def repOnrE(r,B):
```

```
    """
    This gives the rep on the fiber of E^r at the fixed point
       corresponding to B.
    Notice that our E is equal to \mathcal E_0 in [ES].
    This algorithm follows (using some work) from formula
       (4.5) in [ES].
    """
    rep = 1
    ibi=[0,0,0]
    bibi12=[0,0,0]
    for q in [0,1,2]:
        ibi[q] = sum((i*B[q][i] for i in range(len(B[q])) ))
        bibi12[q] = sum((B[q][i]*(B[q][i]-1)/2 for i in range(
           len(B[q])) ))
    for ind in [[0,1,2],[1,2,0],[2,0,1]]:
        rep *=  ( la[ind[0]]**(-ibi[ind[0]]- bibi12[ind[0]]) *
            la[ind[1]]**(ibi[ind[0]]) * la[ind[2]]**(bibi12[
           ind[0]]) )**r
    return rep

repOnSmVdict = dict()
def repOnSmV(m):
    """
    Gives the representation on Sym^m(V), required to compute
       the rep on I_n (see [ES] p188).

    Since this function gets called lots of times with the
       same m, I made a dictionary that stores the results.
    """
    value = repOnSmVdict.get(m,None)
    if value == None:
        #print "recomputing!",m
        value= sum(( prod(( la[i]**expon[i] for i in range(3)
           )) for expon in IntegerVectors(m,3) ))
        repOnSmVdict[m] = value
    return value

def repOnEmx(B,m):
    """Formula 4.5 of [ES]."""
    return sum(( prod(( la[i]**N[i] for i in range(3) )) for N
        in DmB(B,m) ))

def repOnIm(B,m):
    """Formula below 4.6 in [ES]."""
    return repOnSmV(m) - repOnEmx(B,m)
```

```
s1la1 = sum(la[i]**-1 for i in range(3))
s2la1 = la[0]**-1*la[1]**-1 + la[1]**-1*la[2]**-1 + la[0]**-1*
   la[2]**-1
def repOnTangent(B):
    """
    Computes the representation on the tangent space of the
       fixed point corresponding to B.
    Formula 4.7 in [ES].
    Above are some expressions that don't need to be
       recomputed every time.
    """
    m = sum(( sum(B[i]) for i in range(3) ))+2
    repIm = [repOnIm(B,m-i) for i in range(3)]
    repIm_dual = [rep.subs({la[i]:la[i]^-1 for i in range(3)})
        for rep in repIm]
    return 1 - sum(( repIm_dual[i]*repIm[i] for i in range(3)
       )) + s1la1*(repIm_dual[1]*repIm[0] + repIm_dual[2]*
       repIm[1]) - s2la1*repIm_dual[2]*repIm[0]

def get_weights(rep,w):
    """
    Given an element of the representation ring (rep) and a
       vector of weights (w) representing a one parameter
       subgroup,
    returns a list of weights for the action of C^*.
    If it is one dimensional, it returns just the one weight
    """
    #print rep, w
    weights = []
    for coef,expon in zip(rep.coefficients(),rep.exponents()):
        weights += [sum(( w[i]*expon[i] for i in range(3) ))
           ]*int(coef)
    if len(weights) > 1:
        return weights
    return weights[0]

def todd_func(weights):
    """
    Returns a list of todd classes with values from the input
       weights substituted in for the chern roots.

    This is useful when converting todd classes into functions
        via Theorem 2.2 of [ES].
    """
    rank = len(weights)
    dim = rank
    S = PowerSeriesRing(QQ,"t",default_prec = dim+2)
```

```
    t=S.gen()
    Q = prod(( weights[i]*t/(1-exp(-weights[i]*t)) for i in
       range(rank) ))
    #print Q
    #print "hey!"
    coefs = [0]*(rank+1)
    for i,c in zip(Q.exponents(),Q.coefficients()):
        coefs[i]=c
    return coefs

try:
    big_poly_ring
except:
    load("the_rings.sage")

def chi_kr(n,k=deg,r=r):
    """
    On Hilb^n(P^2), computes chi((kH)_n * E^r) (the
       coefficients as in Thm 5.3 of [ESG]) using localization
       .

    You must pass in a value for n. For r and k, you can pass
       in a value, or a variable.
    If r is omitted it will be left as a variable.
    If k is omitted it will be set to the variable deg.

    The weights in the second line can be any triple that does
        not cause a divide by zero error!

    Note:
    A user should just call the Euler function from Euler.sage
       , which will be much faster
    since it loads saved values instead of recomputing them.
    """
    result = 0
    weights = [0,1,19]

    for B in Tripartitions(n):
        line_bundle_weight = k*get_weights(repOnkH(1,B),
           weights) + r*get_weights(repOnrE(1,B),weights)
        tan_weights = get_weights(repOnTangent(B),weights)
        sigma_n_tan = prod(tan_weights)
        todd_funcs = todd_func(tan_weights)
        result += ( todd_funcs[2*n] + sum(( todd_funcs[2*n-i]
           * line_bundle_weight**i/factorial(i) for i in range
           (1,2*n+1) )))/sigma_n_tan
```

```
    return result

def PhiP2(p,k=deg):
    """
    Assembles the generating series of [ESG], Thm 5.3, for P
       ^2, to order p, with k being the number of hyperplane
       classes on P^2
    If k is omitted, it will be treated as a variable named
       deg.
    """
    return (1 + sum(( chi_kr(n,k)*z**n for n in range(1,p+1) )
       )).add_bigoh(p+1)
```

A.3 Euler.sage

This file uses the results of the localization computations to compute the A and B series of Theorem 2.4.1.

```
"""
This file computes the series A and B using the results of the
    Localization computations.
It also contains convenient user functions for accessing the
   Euler characteristics of line bundles on the Hilbert scheme
    of a surface.
"""

try:
    big_power_series_ring
except:
    load("the_rings.sage")
"""
This attempts to loads the saved series from files.
Note: we really need only Phi for the Euler function, but A
   and B are interesting too,
      so we load them in case you want to examine them.
"""
subdir = "LocalizationResults"
try:
    Aseries = load(os.path.join(load_attach_path()[-1],subdir,
       "Afile"))
    Bseries = load(os.path.join(load_attach_path()[-1],subdir,
       "Bfile"))
    Phiseries = load(os.path.join(load_attach_path()[-1],
       subdir,"Phifile"))
except:
    print "There␣was␣an␣error␣loading␣files.␣You␣will␣not␣be␣
       able␣to␣call␣'Euler()'."
```

```
    print "The␣files␣need␣to␣be␣in␣the␣last␣entry␣of␣your␣
       load_attach_path()␣in␣the␣subdirectory␣specified␣by␣
       subdir."
    print "You␣may␣need␣to␣call␣make_PhiAB␣if␣you␣need␣to␣
       recompute␣the␣files."


def Euler(n, KL=KL, K2 = K2, chiL=chiL, r=r, chiS = 1):
    """
    Computes the Euler characteristic as in [EGL] using a
       saved version of A, B.
    Here we are not restricted to P^2.
    You can provide values for K.L, K^2, chi(L), r or leave
       them as variables.
    chiS defaults to 1, but you can supply a variable if you
       like.
    """
    return Phiseries.coefficients()[n].subs(KL=KL, K2 =K2,chiL
       =chiL,chiS=chiS,r=r)


def EulerP2(n, k = deg, r = r):
    """
    Computes the Euler characteristic on P2 when L = kH.
    You can leave k and r as variables or supply them.
    """
    return Euler(n=n, KL=-3*k, K2 = 9,chiL=binomial(k+2,2) ,
       chiS=1,r=r)

def logA(p):
    """
    Computes the logarithm of A to order p. Notice that you
       don't even need f and g to do this!

    Probably this is not optimial speed, since it computes Phi
        twice. There seems to be some difficultly substituting
    values for k if you compute Phi with a variable.
    """
    return RZ( 1/3*(log(Phi(p,-2)) - log(Phi(p,-1))) )

def g(p):
    """g_{1,r^2-1} as in [ESG], just before Lemma 5.2."""
    return (sum(( 1/(1-(Rvar^2-1)*n)*binomial(1 - (Rvar^2-1)*n
       ,n)*RZ.gen()**n for n in range(p+1) ))).add_bigoh(p+1)

def f(p):
```

```
    """f_{0,r^2-1} as in [ESG], just before Lemma 5.2."""
    return (sum(( binomial(-(Rvar^2-1)*(n-1),n)*RZ.gen()**n
       for n in range(p+1) ))).add_bigoh(p+1)

def make_PhiAB(p,path):
    """
    Make and save A,B, and Phi up to order p and saves them in
        the specified path.
    A and B are the series in [EGS], Theorem 5.3.
    Phi is the generating function for Euler characteristics
       in Theorem 5.3.

    This method also returns Phi.

    You only need to run this if you lost the files, or if you
        want to recompute for a bigger p!
    """
    try:
        PhiP2
    except:
        load("Localization.sage")

    logPhim1 = log(PhiP2(p,-1))
    logA = RZ( 1/3*(log(PhiP2(p,-2)) - logPhim1) )
    logf = log(f(p))
    logB = 1/9*(RZ(logPhim1) - 1/2*logf + 3/2*logA)

    logPhi = RS.gen(3)*log(g(p)) + RS.gen(4)/2*logf + (RS.gen
       (2)-RS.gen(1)/2)*logA + RS.gen(1)*logB
    A = exp(logA)
    B = exp(logB)
    Phi = exp(logPhi)
    A.save(os.path.join(path, "Afile"))
    B.save(os.path.join(path, "Bfile"))
    Phi.save(os.path.join(path,"Phifile"))
    return Phi
```

A.4 Q.sage

This file computes the polynomials $\widehat{Q_i}$ discussed in Section 2.7.3.

```
"""
This file computes \hat Q_d, as described in Berczi and Szenes
    "Thom Polynomials of Morin Singularities, section 8.

Unfortunately, this seems to be infeasible above d=6.
```

```
Notice that we use n instead of d in the code.
"""


def Qhat(d,z):
    """
    Returns the function Qhat needed for formula 7.26 of [BS].

    Should work for d <=6.

    The z is a list of variables to be used.
    """
    if d <=3:
        return 1
    else:
        Q = makeQ(d)

        return Q.subs({Q.parent().gen(i) : z[i+1] for i in
           range(n)})


def basicEq(d):
    """
    Returns the initial ideal from the basic equations as in [
       BS], Prop. 8.3.
    Also returns a dictionary that has the weights of the "q"
       variables.

    For d=6, it picks out the correct component and returns
       that ideal.

    For d=>7, it just returns the ideal.
    """
    Ilist = []
    vars = []
    for l in range(1,d+1):
        for m in range(1, l):
            for r in range(m, l-m+1):
                vars.append("q" + str(m) + str(r) + "_" + str(
                   l))

    Sz = PolynomialRing(QQ, ["z" + str(i) for i in range(1,d
       +1)])
    z = [None] + list(Sz.gens())

    R = PolynomialRing(QQ, vars)
    u = dict()
```

```
    weights = dict()

    index = 0
    for l in range(1,d+1):
        for m in range(1, l):
            for r in range(m, l-m+1):
                u[(m,r,l)] = R.gen(index)
                u[(r,m,l)] = R.gen(index)
                weights[R.gen(index)] = z[r] + z[m] - z[l]
                index += 1


    for l in range(1,d+1):
        for sumijm in range(3,l+1):
            for (i,j,m) in Compositions(sumijm, length=3):
                Ilist.append( sum(( u[(j,m,s)]*u[(i,s,l)] for
                    s in range(j+m,l-i+1) )) -
                              sum(( u[(i,m,s)]*u[(j,s,l)] for
                                 s in range(i+m,l-j+1) )) )
    M = R*Ilist

    if d == 6:
        M = M.primary_decomposition()[1]

    lmList = [f.lm() for f in M.groebner_basis()]
    Min = R*lmList

    if d >= 7:
        print "for d=7, returning the ideal (not the initial 
           ideal)."
        return M, weights

    return Min, weights

try:
    save_results
except:
    load("SaveDecorator.sage")

@save_results(os.path.join(load_attach_path()[-1], "QResults")
   )
def makeQ(d):
    """
    Uses the basic equations to make Q. Should work for d<=6.
    """
    M, weights = basicEq(d)
    Q = 0
```

```
    for pi in M.primary_decomposition():
        zmonom = 1
        multiplicity = 1
        for q in pi.gens():
            zmonom *= prod(( weights[q.parent().gen(i)] for i
               in range(q.parent().ngens()) if q.exponents()
               [0][i] >0 ))
            multiplicity *= prod((q.exponents()[0][i] for i in
                range(q.parent().ngens()) if q.exponents()[0][
               i] >0 ))
        Q += multiplicity*zmonom
    return Q
```

A.5 ePTheta.sage

```
"""
This file computes eP[Theta_d] as in Theorem 7.16 of Berczi
   and Szenes "Thom Polynomials of Morin
Singularities. See also section 8 of that paper.

This file is used by MultiplePointFormulas.sage.
"""

try:
    inf_poly_x
except:
    load("the_rings.sage")

try:
    save_results
except:
    load("SaveDecorator.sage")

try:
    Qhat
except:
    load("Q.sage")

@save_results(os.path.join(load_attach_path()[-1], "
   epThetaResults"))
def epTheta(n, mcac=5):

    """
    Computes formula (7.26) in Berczi and Szenes.
    Here, as in MultiplePointFormulas, we have x[k]=c_{d+4-k},
        where d is the codimension (k-n in [BS]).
```

```
    Also we use n for the number of points instead of [BS]'s d
       .
    We are able to eliminate the codimension from the formula
       by using x[k] instead of c_{...},
    since we know some vanishing of our chern classes.

    This is the recursive algorithm, seems to be faster for
       large n, since it cuts out some unnecessary computation
       .

    NOTES:
    We take the residues in the order suggested by the dz_i
       instead of the Res. ???
    We also take the residues at 0 instead of infinity, as it
       must be for the degrees to work out. ???

    This seems to give an answer matching [BS] (8.2) for n =
       2.

    The parameter mcac is the highest non-vanishing chern
       class above the codimension. The default of 5 seems
    to be correct for our application.
    """
    if n == 0:
        return 1

    if n == 1:
        z = [None, var("z1")]
    else:
        z = [None] + list( var(["z" + str(i)  for i in range
            (1,n+1)]) )

    f0top = prod(( prod(( (z[m]-z[l]) for m in range(1,l) ))
       for l in range(1,n+1) ))
    f0bottom = prod(( prod(( prod(( z[m] + z[r] - z[l] for r
       in range(1,min(m,l-m)+1) )) for m in range(1,l) )) for
       l in range(1,n+1) ))
    f0 = Qhat(n,z)*f0top/f0bottom

    return epRec(f0, z, [], n, mcac)

def epRec(f, z, degs, n, mcac):
    """
    The recursive function for epTheta.
    """
    if len(degs) == n:
        return QQ(f)
```

```
    max_degree = mcac - 1
    f2 = taylor(f, z[len(degs) + 1], 0, max_degree)
    Theta_contribution = 0
    for c,e in f2.coefficients():
        if -(e+1-mcac) >= 0:
            Theta_contribution += x[-(e+1 - mcac)]*epRec(c, z,
                degs + [e], n, mcac)
    return Theta_contribution
```

A.6 MultiplePointFormula.sage

This file implements Theorem 2.7.1. It also plugs in the appropriate chern classes of the relative tangent bundle.

```
"""
Executes the plan for computing the cohomology class y_n of
   the closure of the n-fold point locus in Y of the map f:X
   -->Y outlined in section *** of the paper.

Connection to the paper:
1. 'chern' returns the values of the Chern classes of f, as
   computed in [***paper, section??].
2. 'push' returns the push forward f_* of a cohomology class
   in X, using the identities in [***citation].
3. 'multKaz' runs the recursion [***citation] using the
   residual polynomials computed in ePTheta.sage.

Notation:
1. r=rk,L are the first two Chern classes of the sheaf
   orthogonal to I_n, which are denoted r,c in the paper.
2. The residual polynomials computed by ePTheta.sage use the
   variables x[i], which is meant to correspond to the Chern
   class c_{d+4-i}.
3. d = r-1 is used in the comments to refer to the codimension
    of f (called $\ell$ in the paper).
4. K is the pullback to X of the canonical class on S ($\
   kappa$ in the paper)
5. K2, KL, and L2 denote powers and multiplication...

More commenting necessary!
"""

try:
    inf_poly_x
except:
    load("the_rings.sage")
```

```
#inf_poly_series.<t> = PowerSeriesRing(inf_poly_x)

"""
The Euler characteristic of O_{S}(L) arises in several places
   in the computation.
"""
chi = L2/2 - KL/2 + 1

def chern(i, rk = r):
    """
    Return the Chern class c_{d+4-i}, which corresponds to x[i
       ] in inf_poly_x.
    """
    return U(binomial(rk+1,i-2)) + U(K*binomial(rk+1,i-1) + c*
       binomial(rk,i-2))*z + U(2*(KK-6)*binomial(rk+1,i) + cK*
       binomial(rk,i-1) + (chi-(n-1)*(rk-1))*binomial(rk-1,i
       -2))*z^2 + O(z^3)

def binom(a,b):
    """Used to simplify the already extensive last line of
       multByChern."""
    return big_poly_ring(binomial(a,b))

def multByChern(f, i, rk=r):
    """
    Multiply a polynomial f by chern(i). Return c2 instead of
       c*c, cK instead of c*K, and KK instead of K*K.
    """
    f = big_power_series_ring(f)
    f0 = big_power_series_ring(f(z=0))
    f1 = big_power_series_ring(derivative(f,z,1)(z=0))
    return binom(rk+1,i-2)*f + binom(rk+1,i-1)*(f0*K*z + f1(L=
       KL,K=K2)*z^2) + binom(rk,i-2)*(f0*L*z + f1(L=L2,K=KL)*z
       ^2) + (2*(K2-6)*binom(rk+1,i) + KL*binom(rk,i-1) + (chi
       -(n-1)*(rk-1))*binom(rk-1,i-2))*f0*z^2

def subsChern(g, rk = r):
    """
    Convert a polynomial g in an infinite polynomial ring with
        variables var[i] into a polynomial in n,r,c,K by
       substituting chern(i) for var[i].
    """
    if g in ZZ: #return g if g is a constant
        return g
    terms = list(g.polynomial()) #extracts a list of pairs (
       coefficient, monomial)
    f = 0
```

```
    for j in range(len(terms)): #iterate over terms of g
        e = (terms[j][1]).exponents()[0].reversed()
        var = e.nonzero_positions()
        h = terms[j][0]
        for i in range(0,len(var)):
            for k in range(0,e[var[i]]):
                h = multByChern(h, var[i], rk=r)
        f += h
    return f

def push(f, rk=r):
    """
    Compute f_* of a cohomology class on X as in (***citation)
       . The power of z determines how each term pushes
       forward. The second term is just multiplication by L.
    """
    f = expand(big_power_series_ring(f))
    return (chi+rk-1-n*(rk-1))*f(z=0) + derivative(f,z,1)(z=0,
       L=L2,K=KL) + derivative(f,z,2)(z=0)/2


try:
    epTheta
except:
    load("ePTheta.sage")

def multKaz(s, rk=r):
    """
    Kazarian's inductive formula for computing the multiple
       point locus y_s.
    Uses the residual polynomials computed in epTheta.
    We have computed epTheta up to 6, so multKaz works up to s
       =7, the septuple point formula.
    """
    y = [1, push(1,rk)]
    for k in range (2,s+1):
        y.append(0)
        for i in range(1,k+1):
            y[k] += binomial(k-1,i-1) * push(subsChern((-1)^(i
               -1)*factorial(i-1)*epTheta(i-1),rk)) * y[k-i]
    return y[s](n=s,K=-3)/factorial(s)
```

A.7 SaveDecorator.sage

This file creates a function decorator that can be used to have a function automatically save its values so the future calls of the function do not need to redo the computation.

```
def save_results(pathname):
    """
    This is a decorator that you can add to a function.

    pathname is the path you want the files to be save in.

    Then, whenever you call the function, it will check to see
        if it has stored the value
    for those inputs in the dictionary this session. If so, it
        returns the value.
    If not, it will check to see if there is a saved file,
       load it, store the value it in the dictionary, and then
        return it.
    If not, it will actually call the function and compute the
        value, and then store it and save it.
    """
    def save_results(f):

        def f_new(*args):
            value = f_new.values_dict.get(args)

            if value == None:
                arg_str = "_".join((str(arg) for arg in args))
                filename = os.path.join(pathname , f.__name__
                   + arg_str)
                try:
                    #print "Checking for file...", filename
                    value = load(filename)
                except:
                    #print "File not found."
                    value = f(*args)
                    #print "Saving result..."
                    value.save(filename)

                f_new.values_dict[args] = value
            return value

        f_new.values_dict = dict()

        return f_new

    return save_results
```

APPENDIX B

COMPUTATIONS

For our result, we need to know the series $A_r(z)$ and $B_r(z)$ of Theorem 2.4.1 to order 7. Our code, using the techniques explained in Section 2.5, can compute them to order 11 or higher.

$$
\begin{aligned}
A_r(z) = {} & 1 + \Big(-\tfrac{1}{6}r^3 + \tfrac{1}{6}r \Big) z^2 + \Big(\tfrac{17}{40}r^5 - \tfrac{5}{8}r^3 + \tfrac{1}{5}r \Big) z^3 + \Big(-\tfrac{631}{630}r^7 + \tfrac{1}{72}r^6 + \tfrac{88}{45}r^5 - \tfrac{1}{36}r^4 - {} \\
& \tfrac{209}{180}r^3 + \tfrac{1}{72}r^2 + \tfrac{29}{140}r \Big) z^4 + \Big(\tfrac{171215}{72576}r^9 - \tfrac{17}{240}r^8 - \tfrac{69619}{12096}r^7 + \tfrac{7}{40}r^6 + \tfrac{16979}{3456}r^5 - \tfrac{11}{80}r^4 - \tfrac{31259}{18144}r^3 + {} \\
& \tfrac{1}{30}r^2 + \tfrac{13}{63}r \Big) z^5 + \Big(-\tfrac{18684667}{3326400}r^{11} + \tfrac{155581}{604800}r^{10} + \tfrac{597209}{36288}r^9 - \tfrac{1699}{2240}r^8 - \tfrac{5513891}{302400}r^7 + \tfrac{23033}{28800}r^6 + {} \\
& \tfrac{114685}{12096}r^5 - \tfrac{2669}{7560}r^4 - \tfrac{519509}{226800}r^3 + \tfrac{229}{4200}r^2 + \tfrac{281}{1386}r \Big) z^6 + \Big(\tfrac{401297449}{29652480}r^{13} - \tfrac{8914439}{10886400}r^{12} - {} \\
& \tfrac{528153667}{11404800}r^{11} + \tfrac{30585833}{10886400}r^{10} + \tfrac{4352347}{69120}r^9 - \tfrac{13405099}{3628800}r^8 - \tfrac{44899771}{1036800}r^7 + \tfrac{25156259}{10886400}r^6 + \tfrac{817639}{51840}r^5 - {} \\
& \tfrac{1859441}{2721600}r^4 - \tfrac{339287}{118800}r^3 + \tfrac{1433}{18900}r^2 + \tfrac{85}{429}r \Big) z^7 + \Big(-\tfrac{448937586017}{13621608000}r^{15} + \tfrac{545510843}{223534080}r^{14} + {} \\
& \tfrac{200873307991}{1556755200}r^{13} - \tfrac{4578488741}{479001600}r^{12} - \tfrac{41467078967}{199584000}r^{11} + \tfrac{653808103}{43545600}r^{10} + \tfrac{13531857077}{76204800}r^9 - \tfrac{1224530857}{101606400}r^8 - {} \\
& \tfrac{4709184889}{54432000}r^7 + \tfrac{56484901}{10886400}r^6 + \tfrac{119034073}{4989600}r^5 - \tfrac{67641247}{59875200}r^4 - \tfrac{7752863023}{2270268000}r^3 + \tfrac{1123657}{11642400}r^2 + \tfrac{9949}{51480}r \Big) z^8 + {} \\
& \Big(\tfrac{1067238963813721}{13173608448000}r^{17} - \tfrac{4580572146421}{653837184000}r^{16} - \tfrac{1868424788136953}{5230697472000}r^{15} + \tfrac{1008805230793}{32691859200}r^{14} + {} \\
& \tfrac{992904065416223}{1494484992000}r^{13} - \tfrac{101113443793}{1796256000}r^{12} - \tfrac{272962348105847}{402361344000}r^{11} + \tfrac{25120947101}{457228800}r^{10} + \tfrac{60592820342221}{146313216000}r^9 - {} \\
& \tfrac{141153688681}{4572288000}r^8 - \tfrac{4423647350867}{28740096000}r^7 + \tfrac{7144205161}{718502400}r^6 + \tfrac{44170826140139}{1307674368000}r^5 - \tfrac{15354074077}{9081072000}r^4 - {} \\
& \tfrac{36013858967}{9081072000}r^3 + \tfrac{48901}{420420}r^2 + \tfrac{6871}{36465}r \Big) z^9 + \Big(-\tfrac{174424081161393493}{868893574348800}r^{19} + \tfrac{740164886921567}{37661021798400}r^{18} + {} \\
& \tfrac{525222462909510739}{533531142144000}r^{17} - \tfrac{1509433069953143}{15692092416000}r^{16} - \tfrac{32583814614724447}{15692092416000}r^{15} + \tfrac{250692752979619}{1255367393280}r^{14} + {} \\
& \tfrac{16579532320075783}{6725182464000}r^{13} - \tfrac{829530467198351}{3621252096000}r^{12} - \tfrac{4367664389322431}{2414168064000}r^{11} + \tfrac{13899331735211}{87787929600}r^{10} + {} \\
& \tfrac{2045747884459907}{2414168064000}r^9 - \tfrac{40609260170929}{603542016000}r^8 - \tfrac{423655761620591}{1681295616000}r^7 + \tfrac{40245068798999}{2353813862400}r^6 + \tfrac{29792663200381}{653837184000}r^5 - {} \\
& \tfrac{96567618673}{40864824000}r^4 - \tfrac{115983959747}{25729704000}r^3 + \tfrac{6135329}{45405360}r^2 + \tfrac{84883}{461890}r \Big) z^{10} + \Big(\tfrac{116318143051394602 6249}{23222315553259520000}r^{21} - {} \\
& \tfrac{579579679172433911}{10670622842880000}r^{20} - \tfrac{1025339491018420061}{379153559715840}r^{19} + \tfrac{1872735601761139117}{6402373705728000}r^{18} + {} \\
& \tfrac{20416615005951242 8213}{32011868528640000}r^{17} - \tfrac{213613070877362479}{313841848320000}r^{16} - \tfrac{682597201376804469 89}{7908814577664000}r^{15} + \tfrac{562543823508571 27}{62768369664000}r^{14} + {} \\
& \tfrac{1706295556538665571}{230577684480000}r^{13} - \tfrac{106334788965735293}{144850083840000}r^{12} - \tfrac{22048146182184907}{5267275776000}r^{11} + \tfrac{3735287162084767}{9656672256000}r^{10} + {} \\
& \tfrac{1415791391211756841}{898728929280000}r^9 - \tfrac{3417456898636337}{26153487360000}r^8 - \tfrac{7817840089978361}{20175547392000}r^7 + \tfrac{639737824247849}{23538138624000}r^6 + {} \\
& \tfrac{154320545850641}{2605132530000}r^5 - \tfrac{43663729272517}{13894040160000}r^4 - \tfrac{38811107288423}{7699613922000}r^3 + \tfrac{295121623}{1929727800}r^2 + \tfrac{15815}{88179}r \Big) z^{11} + O(z^{12})
\end{aligned}
$$

$$B_r(z) = 1+\left(-\tfrac{1}{24}r^4+\tfrac{1}{24}r^2\right)z^2+\left(\tfrac{97}{720}r^6-\tfrac{31}{144}r^4+\tfrac{29}{360}r^2\right)z^3+\left(-\tfrac{14899}{40320}r^8+\tfrac{2273}{2880}r^6-\tfrac{3053}{5760}r^4+\tfrac{139}{1260}r^2\right)z^4+\left(\tfrac{503377}{518400}r^{10}-\tfrac{311701}{120960}r^8+\tfrac{421267}{172800}r^6-\tfrac{6257}{6480}r^4+\tfrac{187}{1400}r^2\right)z^5+\left(-\tfrac{1205178661}{479001600}r^{12}+\tfrac{346550543}{43545600}r^{10}-\tfrac{19975933}{2073600}r^8+\tfrac{241348529}{43545600}r^6-\tfrac{4092191}{2721600}r^4+\tfrac{9047}{59400}r^2\right)z^6+\left(\tfrac{1571744023}{242161920}r^{14}-\tfrac{11403389887}{479001600}r^{12}+\tfrac{1523544803}{43545600}r^{10}-\tfrac{2666500579}{101606400}r^8+\tfrac{458713229}{43545600}r^6-\tfrac{63757807}{29937600}r^4+\tfrac{6349289}{37837800}r^2\right)z^7+\left(-\tfrac{4667832604837}{278970531840}r^{16}+\tfrac{24333113013161}{348713164800}r^{14}-\tfrac{922918282417}{7664025600}r^{12}+\tfrac{270815028119}{2438553600}r^{10}-\tfrac{573952996279}{9754214400}r^8+\tfrac{4266097853}{239500800}r^6-\tfrac{30868416583}{10897286400}r^4+\tfrac{13689463}{75675600}r^2\right)z^8+\left(\tfrac{3590078272738523}{83147710464000}r^{18}-\tfrac{6329997344747161}{31384184832000}r^{16}+\tfrac{25065806163256451}{62768369664000}r^{14}-\tfrac{451480024671631}{1034643456000}r^{12}+\tfrac{251765287563011}{877879296000}r^{10}-\tfrac{12683520932623}{109734912000}r^8+\tfrac{93306009627791}{3362591232000}r^6-\tfrac{589056916537}{163459296000}r^4+\tfrac{370803619}{1929727800}r^2\right)z^9+\left(-\tfrac{271510360892172250861}{2432902008176640000}r^{20}+\tfrac{147798138481422681 13}{25609494822912000}r^{18}-\tfrac{539229430013624239}{418455797760000}r^{16}+\tfrac{1225036003894071829}{753220435968000}r^{14}-\tfrac{738704283588907679}{579400335360000}r^{12}+\tfrac{917747389224569}{1430618112000}r^{10}-\tfrac{97125031512835757}{470762772480000}r^8+\tfrac{1915375009945933}{47076277248000}r^6-\tfrac{123257763991823}{27788080320000}r^4+\tfrac{7405152103}{36664828200}r^2\right)z^{10}+\left(\tfrac{16241687280917531225036 9}{562000363888803840000}r^{22}-\tfrac{2990401510735699340377}{182467650613248000 0}r^{20}+\tfrac{2483130570212387287 1}{6097498767360000}r^{18}-\tfrac{57602949922460860747}{9886018222080000}r^{16}+\tfrac{17118334582537199287}{3228087582720000}r^{14}-\tfrac{728065935026398553}{227621560320000}r^{12}+\tfrac{20413289987352338423 3}{158176291553280000}r^{10}-\tfrac{967501577627134597}{2824576634880000}r^8+\tfrac{581341871581109}{10207866240000}r^6-\tfrac{2458947330907783}{461976835320000}r^4+\tfrac{84949542443}{403313110200}r^2\right)z^{11}+O(z^{12})$$

As an intermediate step in computing the Thom polynomials $R_{A_i}(\ell-1)$, one needs the auxiliary polynomials $\widehat{Q_d}$. These are given in [BS12] up to $d=5$, but $d=6$ is "too long to present" there. We have no such inhibitions.

$$\widehat{Q_6} = 8z_1^7+44z_1^6z_2+84z_1^5z_2^2+70z_1^4z_2^3+26z_1^3z_2^4+4z_1^2z_2^5+12z_1^6z_3+70z_1^5z_2z_3+136z_1^4z_2^2z_3+103z_1^3z_2^3z_3+29z_1^2z_2^4z_3+2z_1z_2^5z_3+8z_1^5z_3^2+38z_1^4z_2z_3^2+71z_1^3z_2^2z_3^2+46z_1^2z_2^3z_3^2+10z_1z_2^4z_3^2+4z_1^4z_3^3+16z_1^3z_2z_3^3+15z_1^2z_2^2z_3^3+4z_1z_2^3z_3^3-4z_1^6z_4-18z_1^5z_2z_4-10z_1^4z_2^2z_4+14z_1^3z_2^3z_4+14z_1^2z_2^4z_4+4z_1z_2^5z_4+6z_1^5z_3z_4+23z_1^4z_2z_3z_4+17z_1^3z_2^2z_3z_4+19z_1^2z_2^3z_3z_4+11z_1z_2^4z_3z_4+2z_2^5z_3z_4-2z_1^4z_3^2z_4+z_1^3z_2z_3^2z_4+14z_1^2z_2^2z_3^2z_4+13z_1z_2^3z_3^2z_4+4z_2^4z_3^2z_4+4z_1z_2^2z_3^3z_4+2z_2^3z_3^3z_4-8z_1^3z_2^2z_4^2-12z_1^2z_2^3z_4^2-4z_1z_2^4z_4^2-16z_1^3z_2z_3z_4^2-24z_1^2z_2^2z_3z_4^2-12z_1z_2^3z_3z_4^2-2z_2^4z_3z_4^2-8z_1^3z_3^2z_4^2-12z_1^2z_2z_3^2z_4^2-12z_1z_2^2z_3^2z_4^2-4z_2^3z_3^2z_4^2-4z_1z_2z_3^3z_4^2-2z_2^2z_3^3z_4^2-4z_1^4z_4^3-2z_1^3z_2z_4^3+2z_1^2z_2^2z_4^3+6z_1^3z_3z_4^3-z_1^2z_2z_3z_4^3-z_1z_2^2z_3z_4^3-2z_1^2z_3^2z_4^3+z_1z_2z_3^2z_4^3-8z_1^6z_5-32z_1^5z_2z_5-54z_1^4z_2^2z_5-48z_1^3z_2^3z_5-22z_1^2z_2^4z_5-4z_1z_2^5z_5-20z_1^5z_3z_5-80z_1^4z_2z_3z_5-95z_1^3z_2^2z_3z_5-44z_1^2z_2^3z_3z_5-11z_1z_2^4z_3z_5-2z_2^5z_3z_5-16z_1^4z_3^2z_5-56z_1^3z_2z_3^2z_5-48z_1^2z_2^2z_3^2z_5-8z_1z_2^3z_3^2z_5+2z_2^4z_3^2z_5-4z_1^3z_3^3z_5-8z_1^2z_2z_3^3z_5-3z_1z_2^2z_3^3z_5-12z_1^4z_2z_4z_5-22z_1^3z_2^2z_4z_5-14z_1^2z_2^3z_4z_5-4z_1z_2^4z_4z_5+6z_1^3z_2z_3z_4z_5-13z_1^2z_2^2z_3z_4z_5-19z_1z_2^3z_3z_4z_5-6z_2^4z_3z_4z_5-4z_1^2z_2z_3^2z_4z_5-10z_1z_2^2z_3^2z_4z_5-5z_2^3z_3^2z_4z_5-2z_1z_2z_3^3z_4z_5-z_2^2z_3^3z_4z_5+4z_1^4z_4^2z_5+14z_1^3z_2z_4^2z_5+16z_1^2z_2^2z_4^2z_5+6z_1z_2^3z_4^2z_5+6z_1^3z_3z_4^2z_5+17z_1^2z_2z_3z_4^2z_5+16z_1z_2^2z_3z_4^2z_5+5z_2^3z_3z_4^2z_5+10z_1z_2z_3^2z_4^2z_5+$$

$$6z_2^2z_3^2z_4^2z_5+2z_1z_3^3z_4^2z_5+z_2z_3^3z_4^2z_5-4z_1^3z_4^3z_5-2z_1^2z_2z_4^3z_5+2z_1z_2^2z_4^3z_5+6z_1^2z_3z_4^3z_5-z_1z_2z_3z_4^3z_5-$$
$$z_2^2z_3z_4^3z_5-2z_1z_3^2z_4^3z_5+z_2z_3^2z_4^3z_5+4z_1^4z_2z_5^2+10z_1^3z_2^2z_5^2+10z_1^2z_2^3z_5^2+4z_1z_2^4z_5^2+8z_1^4z_3z_5^2+$$
$$18z_1^3z_2z_3z_5^2+11z_1^2z_2^2z_3z_5^2+5z_1z_2^3z_3z_5^2+2z_2^4z_3z_5^2+8z_1^3z_3^2z_5^2+18z_1^2z_2z_3^2z_5^2+z_1z_2^2z_3^2z_5^2-2z_2^3z_3^2z_5^2+$$
$$4z_1^4z_4z_5^2+10z_1^3z_2z_4z_5^2+6z_1^2z_2^2z_4z_5^2-6z_1^3z_3z_4z_5^2-7z_1^2z_2z_3z_4z_5^2+7z_1z_2^2z_3z_4z_5^2+4z_2^3z_3z_4z_5^2+$$
$$2z_1^2z_3^2z_4z_5^2+3z_1z_2z_3^2z_4z_5^2-4z_1^2z_2z_4^2z_5^2-4z_1z_2^2z_4^2z_5^2-4z_1^2z_3z_4^2z_5^2-4z_1z_2z_3z_4^2z_5^2-2z_2^2z_3z_4^2z_5^2-$$
$$2z_2z_3^2z_4^2z_5^2-24z_1^6z_6-112z_1^5z_2z_6-180z_1^4z_2^2z_6-117z_1^3z_2^3z_6-29z_1^2z_2^4z_6-2z_1z_2^5z_6-36z_1^5z_3z_6-$$
$$160z_1^4z_2z_3z_6-233z_1^3z_2^2z_3z_6-129z_1^2z_2^3z_3z_6-23z_1z_2^4z_3z_6-14z_1^4z_3^2z_6-63z_1^3z_2z_3^2z_6-89z_1^2z_2^2z_3^2z_6-$$
$$42z_1z_2^3z_3^2z_6-6z_2^4z_3^2z_6-6z_1^3z_3^3z_6-15z_1^2z_2z_3^3z_6-11z_1z_2^2z_3^3z_6-2z_2^3z_3^3z_6+8z_1^5z_4z_6+24z_1^4z_2z_4z_6+$$
$$14z_1^3z_2^2z_4z_6-12z_1^2z_2^3z_4z_6-11z_1z_2^4z_4z_6-2z_2^5z_4z_6-12z_1^4z_3z_4z_6-16z_1^3z_2z_3z_4z_6-3z_1^2z_2^2z_3z_4z_6-$$
$$7z_1z_2^3z_3z_4z_6-3z_2^4z_3z_4z_6+12z_1^3z_3^2z_4z_6+8z_1^2z_2z_3^2z_4z_6-3z_1z_2^2z_3^2z_4z_6-3z_2^3z_3^2z_4z_6+4z_1z_2z_3^3z_4z_6+$$
$$4z_1^4z_4^2z_6+6z_1^3z_2z_4^2z_6+14z_1^2z_2^2z_4^2z_6+13z_1z_2^3z_4^2z_6+2z_2^4z_4^2z_6-2z_1^3z_3z_4^2z_6+29z_1^2z_2z_3z_4^2z_6+$$
$$26z_1z_2^2z_3z_4^2z_6+6z_2^3z_3z_4^2z_6+14z_1^2z_3^2z_4^2z_6+11z_1z_2z_3^2z_4^2z_6+6z_2^2z_3^2z_4^2z_6+2z_2z_3^3z_4^2z_6+4z_1^3z_4^3z_6+$$
$$2z_1^2z_2z_4^3z_6-2z_1z_2^2z_4^3z_6-6z_1^2z_3z_4^3z_6+z_1z_2z_3z_4^3z_6+z_2^2z_3z_4^3z_6+2z_1z_3^2z_4^3z_6-z_2z_3^2z_4^3z_6+$$
$$32z_1^5z_5z_6+114z_1^4z_2z_5z_6+143z_1^3z_2^2z_5z_6+78z_1^2z_2^3z_5z_6+19z_1z_2^4z_5z_6+2z_2^5z_5z_6+54z_1^4z_3z_5z_6+$$
$$161z_1^3z_2z_3z_5z_6+146z_1^2z_2^2z_3z_5z_6+46z_1z_2^3z_3z_5z_6+5z_2^4z_3z_5z_6+28z_1^3z_3^2z_5z_6+58z_1^2z_2z_3^2z_5z_6+$$
$$38z_1z_2^2z_3^2z_5z_6+4z_2^3z_3^2z_5z_6+6z_1^2z_3^3z_5z_6+7z_1z_2z_3^3z_5z_6+z_2^2z_3^3z_5z_6-8z_1^4z_4z_5z_6-12z_1^3z_2z_4z_5z_6+$$
$$6z_1^2z_2^2z_4z_5z_6+14z_1z_2^3z_4z_5z_6+4z_2^4z_4z_5z_6-2z_1^2z_2z_3z_4z_5z_6+5z_1z_2^2z_3z_4z_5z_6+6z_2^3z_3z_4z_5z_6-$$
$$2z_1^2z_3^2z_4z_5z_6-6z_1z_2z_3^2z_4z_5z_6+2z_2^2z_3^2z_4z_5z_6-2z_1z_3^3z_4z_5z_6-4z_1^3z_4^2z_5z_6-20z_1^2z_2z_4^2z_5z_6-$$
$$20z_1z_2^2z_4^2z_5z_6-4z_2^3z_4^2z_5z_6-16z_1^2z_3z_4^2z_5z_6-22z_1z_2z_3z_4^2z_5z_6-11z_2^2z_3z_4^2z_5z_6-3z_1z_3^2z_4^2z_5z_6-$$
$$8z_2z_3^2z_4^2z_5z_6-z_3^3z_4^2z_5z_6-8z_1^4z_5^2z_6-26z_1^3z_2z_5^2z_6-27z_1^2z_2^2z_5^2z_6-13z_1z_2^3z_5^2z_6-2z_2^4z_5^2z_6-$$
$$18z_1^3z_3z_5^2z_6-33z_1^2z_2z_3z_5^2z_6-15z_1z_2^2z_3z_5^2z_6-5z_2^3z_3z_5^2z_6-14z_1^2z_3^2z_5^2z_6-11z_1z_2z_3^2z_5^2z_6+$$
$$3z_2^2z_3^2z_5^2z_6-4z_1^3z_4z_5^2z_6-2z_1^2z_2z_4z_5^2z_6-z_1z_2^2z_4z_5^2z_6-2z_2^3z_4z_5^2z_6+10z_1^2z_3z_4z_5^2z_6-4z_2^2z_3z_4z_5^2z_6-$$
$$2z_1z_3^2z_4z_5^2z_6+2z_2z_3^2z_4z_5^2z_6+4z_1^2z_4^2z_5^2z_6+7z_1z_2z_4^2z_5^2z_6+2z_2^2z_4^2z_5^2z_6+3z_1z_3z_4^2z_5^2z_6+$$
$$5z_2z_3z_4^2z_5^2z_6+z_3^2z_4^2z_5^2z_6+28z_1^5z_6^2+108z_1^4z_2z_6^2+133z_1^3z_2^2z_6^2+61z_1^2z_2^3z_6^2+9z_1z_2^4z_6^2+36z_1^4z_3z_6^2+$$
$$122z_1^3z_2z_3z_6^2+130z_1^2z_2^2z_3z_6^2+48z_1z_2^3z_3z_6^2+4z_2^4z_3z_6^2+6z_1^3z_3^2z_6^2+37z_1^2z_2z_3^2z_6^2+36z_1z_2^2z_3^2z_6^2+$$
$$10z_2^3z_3^2z_6^2+2z_1^2z_3^3z_6^2+3z_1z_2z_3^3z_6^2+2z_2^2z_3^3z_6^2-4z_1^4z_4z_6^2-10z_1^3z_2z_4z_6^2-10z_1^2z_2^2z_4z_6^2-2z_1z_2^3z_4z_6^2+$$
$$z_2^4z_4z_6^2+2z_1^3z_3z_4z_6^2-17z_1^2z_2z_3z_4z_6^2-14z_1z_2^2z_3z_4z_6^2-2z_2^3z_3z_4z_6^2-14z_1^2z_3^2z_4z_6^2-9z_1z_2z_3^2z_4z_6^2-$$
$$3z_2^2z_3^2z_4z_6^2-2z_2z_3^3z_4z_6^2-4z_1^3z_4^2z_6^2-8z_1^2z_2z_4^2z_6^2-7z_1z_2^2z_4^2z_6^2-3z_2^3z_4^2z_6^2-14z_1z_2z_3z_4^2z_6^2-$$
$$6z_2^2z_3z_4^2z_6^2-6z_1z_3^2z_4^2z_6^2-z_2z_3^2z_4^2z_6^2-42z_1^4z_5z_6^2-115z_1^3z_2z_5z_6^2-104z_1^2z_2^2z_5z_6^2-36z_1z_2^3z_5z_6^2-$$
$$5z_2^4z_5z_6^2-52z_1^3z_3z_5z_6^2-106z_1^2z_2z_3z_5z_6^2-64z_1z_2^2z_3z_5z_6^2-8z_2^3z_3z_5z_6^2-12z_1^2z_3^2z_5z_6^2-$$
$$20z_1z_2z_3^2z_5z_6^2-10z_2^2z_3^2z_5z_6^2-2z_1z_3^3z_5z_6^2-z_2z_3^3z_5z_6^2+12z_1^3z_4z_5z_6^2+18z_1^2z_2z_4z_5z_6^2+4z_1z_2^2z_4z_5z_6^2-$$

$$2z_2^3z_4z_5z_6^2 + 4z_1^2z_3z_4z_5z_6^2 + 17z_1z_2z_3z_4z_5z_6^2 + 6z_2^2z_3z_4z_5z_6^2 + 7z_1z_3^2z_4z_5z_6^2 + 4z_2z_3^2z_4z_5z_6^2 +$$
$$z_3^3z_4z_5z_6^2 + 6z_1^2z_4^2z_5z_6^2 + 12z_1z_2z_4^2z_5z_6^2 + 6z_2^2z_4^2z_5z_6^2 + 7z_1z_3z_4^2z_5z_6^2 + 7z_2z_3z_4^2z_5z_6^2 + 2z_3^2z_4^2z_5z_6^2 +$$
$$14z_1^3z_5^2z_6^2 + 27z_1^2z_2z_5^2z_6^2 + 17z_1z_2^2z_5^2z_6^2 + 5z_2^3z_5^2z_6^2 + 16z_1^2z_3z_5^2z_6^2 + 18z_1z_2z_3z_5^2z_6^2 + 4z_2^2z_3z_5^2z_6^2 +$$
$$6z_1z_3^2z_5^2z_6^2 - z_2z_3^2z_5^2z_6^2 - 4z_1^2z_4z_5^2z_6^2 - 4z_1z_2z_4z_5^2z_6^2 + z_2^2z_4z_5^2z_6^2 - 3z_1z_3z_4z_5^2z_6^2 - 3z_2z_3z_4z_5^2z_6^2 -$$
$$z_3^2z_4z_5^2z_6^2 - 3z_1z_4^2z_5^2z_6^2 - 3z_2z_4^2z_5^2z_6^2 - 2z_3z_4^2z_5^2z_6^2 - 16z_1^4z_6^3 - 48z_1^3z_2z_6^3 - 40z_1^2z_2^2z_6^3 - 10z_1z_2^3z_6^3 -$$
$$12z_1^3z_3z_6^3 - 36z_1^2z_2z_3z_6^3 - 27z_1z_2^2z_3z_6^3 - 6z_2^3z_3z_6^3 - 8z_1z_2z_3^2z_6^3 - 4z_2^2z_3^2z_6^3 + 6z_1^2z_2z_4z_6^3 + 7z_1z_2^2z_4z_6^3 +$$
$$2z_2^3z_4z_6^3 + 6z_1^2z_3z_4z_6^3 + 11z_1z_2z_3z_4z_6^3 + 4z_2^2z_3z_4z_6^3 + 4z_1z_3^2z_4z_6^3 + 2z_2z_3^2z_4z_6^3 + 2z_1z_2z_4^2z_6^3 +$$
$$z_2^2z_4^2z_6^3 + 2z_1z_3z_4^2z_6^3 + z_2z_3z_4^2z_6^3 + 24z_1^3z_5z_6^3 + 48z_1^2z_2z_5z_6^3 + 28z_1z_2^2z_5z_6^3 + 4z_2^3z_5z_6^3 + 18z_1^2z_3z_5z_6^3 +$$
$$23z_1z_2z_3z_5z_6^3 + 7z_2^2z_3z_5z_6^3 + 4z_2z_3^2z_5z_6^3 - 6z_1^2z_4z_5z_6^3 - 10z_1z_2z_4z_5z_6^3 - 4z_2^2z_4z_5z_6^3 - 7z_1z_3z_4z_5z_6^3 -$$
$$6z_2z_3z_4z_5z_6^3 - 2z_3^2z_4z_5z_6^3 - 2z_1z_4^2z_5z_6^3 - 2z_2z_4^2z_5z_6^3 - z_3z_4^2z_5z_6^3 - 8z_1^2z_5^2z_6^3 - 10z_1z_2z_5^2z_6^3 - 4z_2^2z_5^2z_6^3 -$$
$$6z_1z_3z_5^2z_6^3 - z_2z_3z_5^2z_6^3 + 3z_1z_4z_5^2z_6^3 + 2z_2z_4z_5^2z_6^3 + 2z_3z_4z_5^2z_6^3 + z_4^2z_5^2z_6^3 + 4z_1^3z_6^4 + 8z_1^2z_2z_6^4 +$$
$$3z_1z_2^2z_6^4 + 4z_1z_2z_3z_6^4 + 2z_2^2z_3z_6^4 - 2z_1z_2z_4z_6^4 - z_2^2z_4z_6^4 - 2z_1z_3z_4z_6^4 - z_2z_3z_4z_6^4 - 6z_1^2z_5z_6^4 -$$
$$7z_1z_2z_5z_6^4 - z_2^2z_5z_6^4 - 2z_2z_3z_5z_6^4 + 2z_1z_4z_5z_6^4 + 2z_2z_4z_5z_6^4 + z_3z_4z_5z_6^4 + 2z_1z_5^2z_6^4 + z_2z_5^2z_6^4 - z_4z_5^2z_6^4$$

In the course of the multiple point computations, one needs to use the polynomials $R_{A_i}(\ell-1)$, which we give here up to $i = 6$. We were not able to compute $R_{A_7}(\ell-1)$ because that requires $\widehat{Q_7}$. These are the formulas one obtains assuming that all chern classes of degree larger than $\ell+4$ vanish.

$$R_{A_0}(\ell-1) = 1$$

$$R_{A_1}(\ell-1) = c_\ell$$

$$R_{A_2}(\ell-1) = {c_\ell}^2 + c_{\ell+1}c_{\ell-1} + 2\,c_{\ell+2}c_{\ell-2} + 4\,c_{\ell+3}c_{\ell-3} + 8\,c_{\ell+4}c_{\ell-4}$$

$$\begin{aligned} R_{A_3}(\ell-1) = {} & 24\,{c_{\ell+3}}^2c_{\ell-6} + 113\,c_{\ell+4}c_{\ell+3}c_{\ell-7} + 113\,{c_{\ell+4}}^2c_{\ell-8} + 65\,c_{\ell+4}c_{\ell-6}c_{\ell+2} \\ & + {c_\ell}^3 + 3\,c_{\ell+1}c_\ell c_{\ell-1} + 2\,c_{\ell+2}{c_{\ell-1}}^2 + {c_{\ell+1}}^2c_{\ell-2} + 7\,c_{\ell+2}c_\ell c_{\ell-2} + 10\,c_{\ell+3}c_{\ell-1}c_{\ell-2} \\ & + 12\,c_{\ell+4}{c_{\ell-2}}^2 + 5\,c_{\ell+2}c_{\ell+1}c_{\ell-3} + 17\,c_{\ell+3}c_\ell c_{\ell-3} + 26\,c_{\ell+4}c_{\ell-1}c_{\ell-3} + 5\,{c_{\ell+2}}^2c_{\ell-4} \\ & + 14\,c_{\ell+3}c_{\ell+1}c_{\ell-4} + 43\,c_{\ell+4}c_\ell c_{\ell-4} + 24\,c_{\ell+3}c_{\ell+2}c_{\ell-5} + 41\,c_{\ell+4}c_{\ell+1}c_{\ell-5} \end{aligned}$$

$$R_{A_4}(\ell-1) = 176\,{c_{\ell+3}}^3c_{\ell-9} + 1404\,c_{\ell+4}{c_{\ell+3}}^2c_{\ell-10} + 3156\,{c_{\ell+4}}^2c_{\ell+3}c_{\ell-11} + 2104\,{c_{\ell+4}}^3c_{\ell-12} +$$
$$264\,{c_{\ell+3}}^2c_{\ell-8}c_{\ell+2} + 1578\,c_{\ell+4}c_{\ell+3}c_{\ell-9}c_{\ell+2} + 1926\,{c_{\ell+4}}^2c_{\ell-10}c_{\ell+2} + 116\,c_{\ell+3}c_{\ell-7}{c_{\ell+2}}^2 +$$
$$406\,c_{\ell+4}c_{\ell-8}{c_{\ell+2}}^2 + 14\,c_{\ell-6}{c_{\ell+2}}^3 + 164\,{c_{\ell+3}}^2c_{\ell-7}c_{\ell+1} + 1027\,c_{\ell+4}c_{\ell+3}c_{\ell-8}c_{\ell+1} +$$

$$1311\, c_{\ell+4}^2 c_{\ell-9} c_{\ell+1} + 132\, c_{\ell+3} c_{\ell-6} c_{\ell+2} c_{\ell+1} + 493\, c_{\ell+4} c_{\ell-7} c_{\ell+2} c_{\ell+1} + 139\, c_{\ell+4} c_{\ell-6} c_{\ell+1}^2 +$$
$$174\, c_{\ell+3}^2 c_{\ell-6} c_{\ell} + 1002\, c_{\ell+4} c_{\ell+3} c_{\ell-7} c_{\ell} + 1222\, c_{\ell+4}^2 c_{\ell-8} c_{\ell} + 492\, c_{\ell+4} c_{\ell-6} c_{\ell+2} c_{\ell} + c_{\ell}^4 +$$
$$663\, c_{\ell+4} c_{\ell+3} c_{\ell-6} c_{\ell-1} + 831\, c_{\ell+4}^2 c_{\ell-7} c_{\ell-1} + 6\, c_{\ell+1} c_{\ell}^2 c_{\ell-1} + 2\, c_{\ell+1}^2 c_{\ell-1}^2 + 9\, c_{\ell+2} c_{\ell} c_{\ell-1}^2 +$$
$$6\, c_{\ell+3} c_{\ell-1}^3 + 686\, c_{\ell+4}^2 c_{\ell-6} c_{\ell-2} + 4\, c_{\ell+1}^2 c_{\ell} c_{\ell-2} + 16\, c_{\ell+2} c_{\ell}^2 c_{\ell-2} + 17\, c_{\ell+2} c_{\ell+1} c_{\ell-1} c_{\ell-2} +$$
$$53\, c_{\ell+3} c_{\ell} c_{\ell-1} c_{\ell-2} + 54\, c_{\ell+4} c_{\ell-1}^2 c_{\ell-2} + 11\, c_{\ell+2}^2 c_{\ell-2}^2 + 21\, c_{\ell+3} c_{\ell+1} c_{\ell-2}^2 + 76\, c_{\ell+4} c_{\ell} c_{\ell-2}^2 +$$
$$c_{\ell+1}^3 c_{\ell-3} + 23\, c_{\ell+2} c_{\ell+1} c_{\ell} c_{\ell-3} + 46\, c_{\ell+3} c_{\ell}^2 c_{\ell-3} + 16\, c_{\ell+2}^2 c_{\ell-1} c_{\ell-3} + 53\, c_{\ell+3} c_{\ell+1} c_{\ell-1} c_{\ell-3} +$$
$$167\, c_{\ell+4} c_{\ell} c_{\ell-1} c_{\ell-3} + 97\, c_{\ell+3} c_{\ell+2} c_{\ell-2} c_{\ell-3} + 143\, c_{\ell+4} c_{\ell+1} c_{\ell-2} c_{\ell-3} + 62\, c_{\ell+3}^2 c_{\ell-3}^2 +$$
$$124\, c_{\ell+4} c_{\ell+2} c_{\ell-3}^2 + 9\, c_{\ell+2} c_{\ell+1}^2 c_{\ell-4} + 26\, c_{\ell+2}^2 c_{\ell} c_{\ell-4} + 77\, c_{\ell+3} c_{\ell+1} c_{\ell} c_{\ell-4} + 142\, c_{\ell+4} c_{\ell}^2 c_{\ell-4} +$$
$$93\, c_{\ell+3} c_{\ell+2} c_{\ell-1} c_{\ell-4} + 182\, c_{\ell+4} c_{\ell+1} c_{\ell-1} c_{\ell-4} + 99\, c_{\ell+3}^2 c_{\ell-2} c_{\ell-4} + 292\, c_{\ell+4} c_{\ell+2} c_{\ell-2} c_{\ell-4} +$$
$$572\, c_{\ell+4} c_{\ell+3} c_{\ell-3} c_{\ell-4} + 356\, c_{\ell+4}^2 c_{\ell-4}^2 + 21\, c_{\ell+2}^2 c_{\ell+1} c_{\ell-5} + 34\, c_{\ell+3} c_{\ell+1}^2 c_{\ell-5} +$$
$$148\, c_{\ell+3} c_{\ell+2} c_{\ell} c_{\ell-5} + 277\, c_{\ell+4} c_{\ell+1} c_{\ell} c_{\ell-5} + 113\, c_{\ell+3}^2 c_{\ell-1} c_{\ell-5} + 313\, c_{\ell+4} c_{\ell+2} c_{\ell-1} c_{\ell-5} +$$
$$558\, c_{\ell+4} c_{\ell+3} c_{\ell-2} c_{\ell-5} + 616\, c_{\ell+4}^2 c_{\ell-3} c_{\ell-5}$$

$$R_{A_5}(\ell-1) = 7777\, c_{\ell+4}^3 c_{\ell-6}^2 + 1456\, c_{\ell+3}^4 c_{\ell-12} + 17058\, c_{\ell+4} c_{\ell+3}^3 c_{\ell-13} +$$
$$63226\, c_{\ell+4}^2 c_{\ell+3}^2 c_{\ell-14} + 92336\, c_{\ell+4}^3 c_{\ell+3} c_{\ell-15} + 2912\, c_{\ell+3}^3 c_{\ell-11} c_{\ell+2} +$$
$$28292\, c_{\ell+4} c_{\ell+3}^2 c_{\ell-12} c_{\ell+2} + 75278\, c_{\ell+4}^2 c_{\ell+3} c_{\ell-13} c_{\ell+2} + 58220\, c_{\ell+4}^3 c_{\ell-14} c_{\ell+2} +$$
$$1976\, c_{\ell+3}^2 c_{\ell-10} c_{\ell+2}^2 + 14542\, c_{\ell+4} c_{\ell+3} c_{\ell-11} c_{\ell+2}^2 + 21184\, c_{\ell+4}^2 c_{\ell-12} c_{\ell+2}^2 +$$
$$520\, c_{\ell+3} c_{\ell-9} c_{\ell+2}^3 + 2268\, c_{\ell+4} c_{\ell-10} c_{\ell+2}^3 + 42\, c_{\ell-8} c_{\ell+2}^4 + 46168\, c_{\ell+4}^4 c_{\ell-16} +$$
$$1872\, c_{\ell+3}^3 c_{\ell-10} c_{\ell+1} + 18764\, c_{\ell+4} c_{\ell+3}^2 c_{\ell-11} c_{\ell+1} + 51604\, c_{\ell+4}^2 c_{\ell+3} c_{\ell-12} c_{\ell+1} +$$
$$41162\, c_{\ell+4}^3 c_{\ell-13} c_{\ell+1} + 2392\, c_{\ell+3}^2 c_{\ell-9} c_{\ell+2} c_{\ell+1} + 18328\, c_{\ell+4} c_{\ell+3} c_{\ell-10} c_{\ell+2} c_{\ell+1} +$$
$$27826\, c_{\ell+4}^2 c_{\ell-11} c_{\ell+2} c_{\ell+1} + 872\, c_{\ell+3} c_{\ell-8} c_{\ell+2}^2 c_{\ell+1} + 4016\, c_{\ell+4} c_{\ell-9} c_{\ell+2}^2 c_{\ell+1} +$$
$$84\, c_{\ell-7} c_{\ell+2}^3 c_{\ell+1} + 681\, c_{\ell+3}^2 c_{\ell-8} c_{\ell+1}^2 + 5494\, c_{\ell+4} c_{\ell+3} c_{\ell-9} c_{\ell+1}^2 + 8766\, c_{\ell+4}^2 c_{\ell-10} c_{\ell+1}^2 +$$
$$449\, c_{\ell+3} c_{\ell-7} c_{\ell+2} c_{\ell+1}^2 + 2222\, c_{\ell+4} c_{\ell-8} c_{\ell+2} c_{\ell+1}^2 + 56\, c_{\ell-6} c_{\ell+2}^2 c_{\ell+1}^2 + 69\, c_{\ell+3} c_{\ell-6} c_{\ell+1}^3 +$$
$$377\, c_{\ell+4} c_{\ell-7} c_{\ell+1}^3 + 1880\, c_{\ell+3}^3 c_{\ell-9} c_{\ell} + 17796\, c_{\ell+4} c_{\ell+3}^2 c_{\ell-10} c_{\ell} + 47322\, c_{\ell+4}^2 c_{\ell+3} c_{\ell-11} c_{\ell} +$$
$$37032\, c_{\ell+4}^3 c_{\ell-12} c_{\ell} + 2470\, c_{\ell+3}^2 c_{\ell-8} c_{\ell+2} c_{\ell} + 17586\, c_{\ell+4} c_{\ell+3} c_{\ell-9} c_{\ell+2} c_{\ell} +$$
$$25570\, c_{\ell+4}^2 c_{\ell-10} c_{\ell+2} c_{\ell} + 942\, c_{\ell+3} c_{\ell-7} c_{\ell+2}^2 c_{\ell} + 3934\, c_{\ell+4} c_{\ell-8} c_{\ell+2}^2 c_{\ell} + 98\, c_{\ell-6} c_{\ell+2}^3 c_{\ell} +$$
$$1405\, c_{\ell+3}^2 c_{\ell-7} c_{\ell+1} c_{\ell} + 10485\, c_{\ell+4} c_{\ell+3} c_{\ell-8} c_{\ell+1} c_{\ell} + 15971\, c_{\ell+4}^2 c_{\ell-9} c_{\ell+1} c_{\ell} +$$
$$975\, c_{\ell+3} c_{\ell-6} c_{\ell+2} c_{\ell+1} c_{\ell} + 4343\, c_{\ell+4} c_{\ell-7} c_{\ell+2} c_{\ell+1} c_{\ell} + 1102\, c_{\ell+4} c_{\ell-6} c_{\ell+1}^2 c_{\ell} + 736\, c_{\ell+3}^2 c_{\ell-6} c_{\ell}^2 +$$
$$5036\, c_{\ell+4} c_{\ell+3} c_{\ell-7} c_{\ell}^2 + 7290\, c_{\ell+4}^2 c_{\ell-8} c_{\ell}^2 + 2144\, c_{\ell+4} c_{\ell-6} c_{\ell+2} c_{\ell}^2 + c_{\ell}^5 + 1290\, c_{\ell+3}^3 c_{\ell-8} c_{\ell-1} +$$
$$12366\, c_{\ell+4} c_{\ell+3}^2 c_{\ell-9} c_{\ell-1} + 33482\, c_{\ell+4}^2 c_{\ell+3} c_{\ell-10} c_{\ell-1} + 26742\, c_{\ell+4}^3 c_{\ell-11} c_{\ell-1} +$$
$$1651\, c_{\ell+3}^2 c_{\ell-7} c_{\ell+2} c_{\ell-1} + 11879\, c_{\ell+4} c_{\ell+3} c_{\ell-8} c_{\ell+2} c_{\ell-1} + 17583\, c_{\ell+4}^2 c_{\ell-9} c_{\ell+2} c_{\ell-1} +$$

$$615\, c_{\ell+3}c_{\ell-6}c_{\ell+2}^2c_{\ell-1} + 2583\, c_{\ell+4}c_{\ell-7}c_{\ell+2}^2c_{\ell-1} + 957\, c_{\ell+3}^2c_{\ell-6}c_{\ell+1}c_{\ell-1} +$$

$$7090\, c_{\ell+4}c_{\ell+3}c_{\ell-7}c_{\ell+1}c_{\ell-1} + 10882\, c_{\ell+4}^2c_{\ell-8}c_{\ell+1}c_{\ell-1} + 2890\, c_{\ell+4}c_{\ell-6}c_{\ell+2}c_{\ell+1}c_{\ell-1} +$$

$$6456\, c_{\ell+4}c_{\ell+3}c_{\ell-6}c_{\ell}c_{\ell-1} + 9544\, c_{\ell+4}^2c_{\ell-7}c_{\ell}c_{\ell-1} + 10\, c_{\ell+1}c_{\ell}^3c_{\ell-1} + 3126\, c_{\ell+4}^2c_{\ell-6}c_{\ell-1}^2 +$$

$$10\, c_{\ell+1}^2c_{\ell}c_{\ell-1}^2 + 25\, c_{\ell+2}c_{\ell}^2c_{\ell-1}^2 + 12\, c_{\ell+2}c_{\ell+1}c_{\ell-1}^3 + 38\, c_{\ell+3}c_{\ell}c_{\ell-1}^3 + 24\, c_{\ell+4}c_{\ell-1}^4 +$$

$$1097\, c_{\ell+3}^3c_{\ell-7}c_{\ell-2} + 10261\, c_{\ell+4}c_{\ell+3}^2c_{\ell-8}c_{\ell-2} + 27475\, c_{\ell+4}^2c_{\ell+3}c_{\ell-9}c_{\ell-2} +$$

$$21886\, c_{\ell+4}^3c_{\ell-10}c_{\ell-2} + 1511\, c_{\ell+3}^2c_{\ell-6}c_{\ell+2}c_{\ell-2} + 10291\, c_{\ell+4}c_{\ell+3}c_{\ell-7}c_{\ell+2}c_{\ell-2} +$$

$$14766\, c_{\ell+4}^2c_{\ell-8}c_{\ell+2}c_{\ell-2} + 2414\, c_{\ell+4}c_{\ell-6}c_{\ell+2}^2c_{\ell-2} + 5629\, c_{\ell+4}c_{\ell+3}c_{\ell-6}c_{\ell+1}c_{\ell-2} +$$

$$8565\, c_{\ell+4}^2c_{\ell-7}c_{\ell+1}c_{\ell-2} + 7688\, c_{\ell+4}^2c_{\ell-6}c_{\ell}c_{\ell-2} + 10\, c_{\ell+1}^2c_{\ell}^2c_{\ell-2} + 30\, c_{\ell+2}c_{\ell}^3c_{\ell-2} +$$

$$5\, c_{\ell+1}^3c_{\ell-1}c_{\ell-2} + 95\, c_{\ell+2}c_{\ell+1}c_{\ell}c_{\ell-1}c_{\ell-2} + 170\, c_{\ell+3}c_{\ell}^2c_{\ell-1}c_{\ell-2} + 39\, c_{\ell+2}^2c_{\ell-1}^2c_{\ell-2} +$$

$$115\, c_{\ell+3}c_{\ell+1}c_{\ell-1}^2c_{\ell-2} + 400\, c_{\ell+4}c_{\ell}c_{\ell-1}^2c_{\ell-2} + 19\, c_{\ell+2}c_{\ell+1}^2c_{\ell-2}^2 + 68\, c_{\ell+2}^2c_{\ell}c_{\ell-2}^2 +$$

$$136\, c_{\ell+3}c_{\ell+1}c_{\ell}c_{\ell-2}^2 + 285\, c_{\ell+4}c_{\ell}^2c_{\ell-2}^2 + 233\, c_{\ell+3}c_{\ell+2}c_{\ell-1}c_{\ell-2}^2 + 389\, c_{\ell+4}c_{\ell+1}c_{\ell-1}c_{\ell-2}^2 +$$

$$78\, c_{\ell+3}^2c_{\ell-2}^3 + 268\, c_{\ell+4}c_{\ell+2}c_{\ell-2}^3 + 1211\, c_{\ell+3}^3c_{\ell-6}c_{\ell-3} + 10081\, c_{\ell+4}c_{\ell+3}^2c_{\ell-7}c_{\ell-3} +$$

$$24933\, c_{\ell+4}^2c_{\ell+3}c_{\ell-8}c_{\ell-3} + 18907\, c_{\ell+4}^3c_{\ell-9}c_{\ell-3} + 9332\, c_{\ell+4}c_{\ell+3}c_{\ell-6}c_{\ell+2}c_{\ell-3} +$$

$$12470\, c_{\ell+4}^2c_{\ell-7}c_{\ell+2}c_{\ell-3} + 7482\, c_{\ell+4}^2c_{\ell-6}c_{\ell+1}c_{\ell-3} + 5\, c_{\ell+1}^3c_{\ell}c_{\ell-3} + 65\, c_{\ell+2}c_{\ell+1}c_{\ell}^2c_{\ell-3} +$$

$$100\, c_{\ell+3}c_{\ell}^3c_{\ell-3} + 41\, c_{\ell+2}c_{\ell+1}^2c_{\ell-1}c_{\ell-3} + 99\, c_{\ell+2}^2c_{\ell}c_{\ell-1}c_{\ell-3} + 344\, c_{\ell+3}c_{\ell+1}c_{\ell}c_{\ell-1}c_{\ell-3} +$$

$$632\, c_{\ell+4}c_{\ell}^2c_{\ell-1}c_{\ell-3} + 229\, c_{\ell+3}c_{\ell+2}c_{\ell-1}^2c_{\ell-3} + 453\, c_{\ell+4}c_{\ell+1}c_{\ell-1}^2c_{\ell-3} + 98\, c_{\ell+2}^2c_{\ell+1}c_{\ell-2}c_{\ell-3} +$$

$$130\, c_{\ell+3}c_{\ell+1}^2c_{\ell-2}c_{\ell-3} + 689\, c_{\ell+3}c_{\ell+2}c_{\ell}c_{\ell-2}c_{\ell-3} + 1095\, c_{\ell+4}c_{\ell+1}c_{\ell}c_{\ell-2}c_{\ell-3} +$$

$$604\, c_{\ell+3}^2c_{\ell-1}c_{\ell-2}c_{\ell-3} + 1550\, c_{\ell+4}c_{\ell+2}c_{\ell-1}c_{\ell-2}c_{\ell-3} + 1580\, c_{\ell+4}c_{\ell+3}c_{\ell-2}^2c_{\ell-3} + 29\, c_{\ell+2}^3c_{\ell-3}^2 +$$

$$302\, c_{\ell+3}c_{\ell+2}c_{\ell+1}c_{\ell-3}^2 + 262\, c_{\ell+4}c_{\ell+1}^2c_{\ell-3}^2 + 499\, c_{\ell+3}^2c_{\ell}c_{\ell-3}^2 + 1044\, c_{\ell+4}c_{\ell+2}c_{\ell}c_{\ell-3}^2 +$$

$$1850\, c_{\ell+4}c_{\ell+3}c_{\ell-1}c_{\ell-3}^2 + 1912\, c_{\ell+4}^2c_{\ell-2}c_{\ell-3}^2 + 8958\, c_{\ell+4}c_{\ell+3}^2c_{\ell-6}c_{\ell-4} +$$

$$23948\, c_{\ell+4}^2c_{\ell+3}c_{\ell-7}c_{\ell-4} + 19052\, c_{\ell+4}^3c_{\ell-8}c_{\ell-4} + 12686\, c_{\ell+4}^2c_{\ell-6}c_{\ell+2}c_{\ell-4} + c_{\ell+1}^4c_{\ell-4} +$$

$$51\, c_{\ell+2}c_{\ell+1}^2c_{\ell}c_{\ell-4} + 82\, c_{\ell+2}^2c_{\ell}^2c_{\ell-4} + 254\, c_{\ell+3}c_{\ell+1}c_{\ell}^2c_{\ell-4} + 366\, c_{\ell+4}c_{\ell}^3c_{\ell-4} +$$

$$93\, c_{\ell+2}^2c_{\ell+1}c_{\ell-1}c_{\ell-4} + 166\, c_{\ell+3}c_{\ell+1}^2c_{\ell-1}c_{\ell-4} + 666\, c_{\ell+3}c_{\ell+2}c_{\ell}c_{\ell-1}c_{\ell-4} +$$

$$1401\, c_{\ell+4}c_{\ell+1}c_{\ell}c_{\ell-1}c_{\ell-4} + 304\, c_{\ell+3}^2c_{\ell-1}^2c_{\ell-4} + 873\, c_{\ell+4}c_{\ell+2}c_{\ell-1}^2c_{\ell-4} + 78\, c_{\ell+2}^3c_{\ell-2}c_{\ell-4} +$$

$$597\, c_{\ell+3}c_{\ell+2}c_{\ell+1}c_{\ell-2}c_{\ell-4} + 579\, c_{\ell+4}c_{\ell+1}^2c_{\ell-2}c_{\ell-4} + 802\, c_{\ell+3}^2c_{\ell}c_{\ell-2}c_{\ell-4} +$$

$$2456\, c_{\ell+4}c_{\ell+2}c_{\ell}c_{\ell-2}c_{\ell-4} + 3657\, c_{\ell+4}c_{\ell+3}c_{\ell-1}c_{\ell-2}c_{\ell-4} + 2270\, c_{\ell+4}^2c_{\ell-2}^2c_{\ell-4} +$$

$$556\, c_{\ell+3}c_{\ell+2}^2c_{\ell-3}c_{\ell-4} + 849\, c_{\ell+3}^2c_{\ell+1}c_{\ell-3}c_{\ell-4} + 2168\, c_{\ell+4}c_{\ell+2}c_{\ell+1}c_{\ell-3}c_{\ell-4} +$$

$$5426\, c_{\ell+4}c_{\ell+3}c_{\ell}c_{\ell-3}c_{\ell-4} + 4977\, c_{\ell+4}^2c_{\ell-1}c_{\ell-3}c_{\ell-4} + 651\, c_{\ell+3}^2c_{\ell+2}c_{\ell-4}^2 +$$

$$1017\, c_{\ell+4}c_{\ell+2}^2c_{\ell-4}^2 + 2706\, c_{\ell+4}c_{\ell+3}c_{\ell+1}c_{\ell-4}^2 + 3916\, c_{\ell+4}^2c_{\ell}c_{\ell-4}^2 + 21026\, c_{\ell+4}^2c_{\ell+3}c_{\ell-6}c_{\ell-5} +$$

$$16070\, c_{\ell+4}^3c_{\ell-7}c_{\ell-5} + 14\, c_{\ell+2}c_{\ell+1}^3c_{\ell-5} + 133\, c_{\ell+2}^2c_{\ell+1}c_{\ell}c_{\ell-5} + 226\, c_{\ell+3}c_{\ell+1}^2c_{\ell}c_{\ell-5} +$$

$$542\, c_{\ell+3}c_{\ell+2}c_{\ell}^2c_{\ell-5} + 1087\, c_{\ell+4}c_{\ell+1}c_{\ell}^2c_{\ell-5} + 63\, c_{\ell+2}^3c_{\ell-1}c_{\ell-5} + 664\, c_{\ell+3}c_{\ell+2}c_{\ell+1}c_{\ell-1}c_{\ell-5} +$$

$$758\,c_{\ell+4}{c_{\ell+1}}^2c_{\ell-1}c_{\ell-5} + 930\,{c_{\ell+3}}^2c_\ell c_{\ell-1}c_{\ell-5} + 2659\,c_{\ell+4}c_{\ell+2}c_\ell c_{\ell-1}c_{\ell-5} +$$
$$2098\,c_{\ell+4}c_{\ell+3}{c_{\ell-1}}^2c_{\ell-5} + 635\,c_{\ell+3}{c_{\ell+2}}^2c_{\ell-2}c_{\ell-5} + 773\,{c_{\ell+3}}^2c_{\ell+1}c_{\ell-2}c_{\ell-5} +$$
$$2459\,c_{\ell+4}c_{\ell+2}c_{\ell+1}c_{\ell-2}c_{\ell-5} + 5331\,c_{\ell+4}c_{\ell+3}c_\ell c_{\ell-2}c_{\ell-5} + 5084\,{c_{\ell+4}}^2c_{\ell-1}c_{\ell-2}c_{\ell-5} +$$
$$1519\,{c_{\ell+3}}^2c_{\ell+2}c_{\ell-3}c_{\ell-5} + 1944\,c_{\ell+4}{c_{\ell+2}}^2c_{\ell-3}c_{\ell-5} + 5414\,c_{\ell+4}c_{\ell+3}c_{\ell+1}c_{\ell-3}c_{\ell-5} +$$
$$6839\,{c_{\ell+4}}^2c_\ell c_{\ell-3}c_{\ell-5} + 972\,{c_{\ell+3}}^3c_{\ell-4}c_{\ell-5} + 8772\,c_{\ell+4}c_{\ell+3}c_{\ell+2}c_{\ell-4}c_{\ell-5} +$$
$$7820\,{c_{\ell+4}}^2c_{\ell+1}c_{\ell-4}c_{\ell-5} + 4164\,c_{\ell+4}{c_{\ell+3}}^2{c_{\ell-5}}^2 + 5713\,{c_{\ell+4}}^2c_{\ell+2}{c_{\ell-5}}^2$$

$$R_{A_6}(\ell-1) = 5652\,{c_{\ell+3}}^4{c_{\ell-6}}^2 + 129621\,c_{\ell+4}{c_{\ell+3}}^3c_{\ell-6}c_{\ell-7} + 262061\,{c_{\ell+4}}^2{c_{\ell+3}}^2{c_{\ell-7}}^2 +$$
$$506958\,{c_{\ell+4}}^2{c_{\ell+3}}^2c_{\ell-6}c_{\ell-8} + 823687\,{c_{\ell+4}}^3c_{\ell+3}c_{\ell-7}c_{\ell-8} + 233451\,{c_{\ell+4}}^4{c_{\ell-8}}^2 +$$
$$814819\,{c_{\ell+4}}^3c_{\ell+3}c_{\ell-6}c_{\ell-9} + 440987\,{c_{\ell+4}}^4c_{\ell-7}c_{\ell-9} + 458298\,{c_{\ell+4}}^4c_{\ell-6}c_{\ell-10} +$$
$$13056\,{c_{\ell+3}}^5c_{\ell-15} + 95732\,c_{\ell+4}{c_{\ell+3}}^2{c_{\ell-6}}^2c_{\ell+2} + 551068\,{c_{\ell+4}}^2c_{\ell+3}c_{\ell-6}c_{\ell-7}c_{\ell+2} +$$
$$225119\,{c_{\ell+4}}^3{c_{\ell-7}}^2c_{\ell+2} + 475906\,{c_{\ell+4}}^3c_{\ell-6}c_{\ell-8}c_{\ell+2} + 32640\,{c_{\ell+3}}^4c_{\ell-14}c_{\ell+2} +$$
$$452420\,c_{\ell+4}{c_{\ell+3}}^3c_{\ell-15}c_{\ell+2} + 72676\,{c_{\ell+4}}^2{c_{\ell-6}}^2{c_{\ell+2}}^2 + 30080\,{c_{\ell+3}}^3c_{\ell-13}{c_{\ell+2}}^2 +$$
$$348288\,c_{\ell+4}{c_{\ell+3}}^2c_{\ell-14}{c_{\ell+2}}^2 + 1079004\,{c_{\ell+4}}^2c_{\ell+3}c_{\ell-15}{c_{\ell+2}}^2 + 12480\,{c_{\ell+3}}^2c_{\ell-12}{c_{\ell+2}}^3 +$$
$$110332\,c_{\ell+4}c_{\ell+3}c_{\ell-13}{c_{\ell+2}}^3 + 188320\,{c_{\ell+4}}^2c_{\ell-14}{c_{\ell+2}}^3 + 2248\,c_{\ell+3}c_{\ell-11}{c_{\ell+2}}^4 +$$
$$11880\,c_{\ell+4}c_{\ell-12}{c_{\ell+2}}^4 + 132\,c_{\ell-10}{c_{\ell+2}}^5 + 207080\,c_{\ell+4}{c_{\ell+3}}^4c_{\ell-16} +$$
$$1941180\,{c_{\ell+4}}^2{c_{\ell+3}}^2c_{\ell+2}c_{\ell-16} + 955576\,{c_{\ell+4}}^3{c_{\ell+2}}^2c_{\ell-16} + 1107800\,{c_{\ell+4}}^2{c_{\ell+3}}^3c_{\ell-17} +$$
$$3230400\,{c_{\ell+4}}^3c_{\ell+3}c_{\ell+2}c_{\ell-17} + 2621520\,{c_{\ell+4}}^3{c_{\ell+3}}^2c_{\ell-18} + 1818160\,{c_{\ell+4}}^4c_{\ell+2}c_{\ell-18} +$$
$$2824480\,{c_{\ell+4}}^4c_{\ell+3}c_{\ell-19} + 1129792\,{c_{\ell+4}}^5c_{\ell-20} + 168654\,{c_{\ell+4}}^2c_{\ell+3}{c_{\ell-6}}^2c_{\ell+1} +$$
$$296252\,{c_{\ell+4}}^3c_{\ell-6}c_{\ell-7}c_{\ell+1} + 21440\,{c_{\ell+3}}^4c_{\ell-13}c_{\ell+1} + 303914\,c_{\ell+4}{c_{\ell+3}}^3c_{\ell-14}c_{\ell+1} +$$
$$1337682\,{c_{\ell+4}}^2{c_{\ell+3}}^2c_{\ell-15}c_{\ell+1} + 37760\,{c_{\ell+3}}^3c_{\ell-12}c_{\ell+2}c_{\ell+1} + 449484\,c_{\ell+4}{c_{\ell+3}}^2c_{\ell-13}c_{\ell+2}c_{\ell+1} +$$
$$1435974\,{c_{\ell+4}}^2c_{\ell+3}c_{\ell-14}c_{\ell+2}c_{\ell+1} + 1309936\,{c_{\ell+4}}^3c_{\ell-15}c_{\ell+2}c_{\ell+1} + 22208\,{c_{\ell+3}}^2c_{\ell-11}{c_{\ell+2}}^2c_{\ell+1} +$$
$$203350\,c_{\ell+4}c_{\ell+3}c_{\ell-12}{c_{\ell+2}}^2c_{\ell+1} + 360468\,{c_{\ell+4}}^2c_{\ell-13}{c_{\ell+2}}^2c_{\ell+1} + 4960\,c_{\ell+3}c_{\ell-10}{c_{\ell+2}}^3c_{\ell+1} +$$
$$27452\,c_{\ell+4}c_{\ell-11}{c_{\ell+2}}^3c_{\ell+1} + 330\,c_{\ell-9}{c_{\ell+2}}^4c_{\ell+1} + 2282488\,{c_{\ell+4}}^3c_{\ell+3}c_{\ell-16}c_{\ell+1} +$$
$$1315000\,{c_{\ell+4}}^4c_{\ell-17}c_{\ell+1} + 11348\,{c_{\ell+3}}^3c_{\ell-11}{c_{\ell+1}}^2 + 139724\,c_{\ell+4}{c_{\ell+3}}^2c_{\ell-12}{c_{\ell+1}}^2 +$$
$$462588\,{c_{\ell+4}}^2c_{\ell+3}c_{\ell-13}{c_{\ell+1}}^2 + 436466\,{c_{\ell+4}}^3c_{\ell-14}{c_{\ell+1}}^2 + 12478\,{c_{\ell+3}}^2c_{\ell-10}c_{\ell+2}{c_{\ell+1}}^2 +$$
$$119356\,c_{\ell+4}c_{\ell+3}c_{\ell-11}c_{\ell+2}{c_{\ell+1}}^2 + 221214\,{c_{\ell+4}}^2c_{\ell-12}c_{\ell+2}{c_{\ell+1}}^2 + 3826\,c_{\ell+3}c_{\ell-9}{c_{\ell+2}}^2{c_{\ell+1}}^2 +$$
$$22460\,c_{\ell+4}c_{\ell-10}{c_{\ell+2}}^2{c_{\ell+1}}^2 + 300\,c_{\ell-8}{c_{\ell+2}}^3{c_{\ell+1}}^2 + 2185\,{c_{\ell+3}}^2c_{\ell-9}{c_{\ell+1}}^3 +$$
$$22094\,c_{\ell+4}c_{\ell+3}c_{\ell-10}{c_{\ell+1}}^3 + 43202\,{c_{\ell+4}}^2c_{\ell-11}{c_{\ell+1}}^3 + 1199\,c_{\ell+3}c_{\ell-8}c_{\ell+2}{c_{\ell+1}}^3 +$$
$$7604\,c_{\ell+4}c_{\ell-9}c_{\ell+2}{c_{\ell+1}}^3 + 120\,c_{\ell-7}{c_{\ell+2}}^2{c_{\ell+1}}^3 + 125\,c_{\ell+3}c_{\ell-7}{c_{\ell+1}}^4 + 881\,c_{\ell+4}c_{\ell-8}{c_{\ell+1}}^4 +$$
$$20\,c_{\ell-6}c_{\ell+2}{c_{\ell+1}}^4 + 132026\,{c_{\ell+4}}^3{c_{\ell-6}}^2c_\ell + 20936\,{c_{\ell+3}}^4c_{\ell-12}c_\ell + 284244\,c_{\ell+4}{c_{\ell+3}}^3c_{\ell-13}c_\ell +$$

$$1217780\,{c_{\ell+4}}^2{c_{\ell+3}}^2c_{\ell-14}c_\ell + 2044096\,{c_{\ell+4}}^3c_{\ell+3}c_{\ell-15}c_\ell + 37480\,{c_{\ell+3}}^3c_{\ell-11}c_{\ell+2}c_\ell +$$
$$423304\,c_{\ell+4}{c_{\ell+3}}^2c_{\ell-12}c_{\ell+2}c_\ell + 1308328\,{c_{\ell+4}}^2c_{\ell+3}c_{\ell-13}c_{\ell+2}c_\ell + 1169704\,{c_{\ell+4}}^3c_{\ell-14}c_{\ell+2}c_\ell +$$
$$22600\,{c_{\ell+3}}^2c_{\ell-10}{c_{\ell+2}}^2c_\ell + 193788\,c_{\ell+4}c_{\ell+3}c_{\ell-11}{c_{\ell+2}}^2c_\ell + 329536\,{c_{\ell+4}}^2c_{\ell-12}{c_{\ell+2}}^2c_\ell +$$
$$5248\,c_{\ell+3}c_{\ell-9}{c_{\ell+2}}^3c_\ell + 26688\,c_{\ell+4}c_{\ell-10}{c_{\ell+2}}^3c_\ell + 372\,c_{\ell-8}{c_{\ell+2}}^4c_\ell + 1166600\,{c_{\ell+4}}^4c_{\ell-16}c_\ell +$$
$$22444\,{c_{\ell+3}}^3c_{\ell-10}c_{\ell+1}c_\ell + 261584\,c_{\ell+4}{c_{\ell+3}}^2c_{\ell-11}c_{\ell+1}c_\ell + 836422\,{c_{\ell+4}}^2c_{\ell+3}c_{\ell-12}c_{\ell+1}c_\ell +$$
$$772644\,{c_{\ell+4}}^3c_{\ell-13}c_{\ell+1}c_\ell + 25360\,{c_{\ell+3}}^2c_{\ell-9}c_{\ell+2}c_{\ell+1}c_\ell + 226380\,c_{\ell+4}c_{\ell+3}c_{\ell-10}c_{\ell+2}c_{\ell+1}c_\ell +$$
$$401464\,{c_{\ell+4}}^2c_{\ell-11}c_{\ell+2}c_{\ell+1}c_\ell + 8120\,c_{\ell+3}c_{\ell-8}{c_{\ell+2}}^2c_{\ell+1}c_\ell + 43568\,c_{\ell+4}c_{\ell-9}{c_{\ell+2}}^2c_{\ell+1}c_\ell +$$
$$684\,c_{\ell-7}{c_{\ell+2}}^3c_{\ell+1}c_\ell + 6647\,{c_{\ell+3}}^2c_{\ell-8}{c_{\ell+1}}^2c_\ell + 62490\,c_{\ell+4}c_{\ell+3}c_{\ell-9}{c_{\ell+1}}^2c_\ell +$$
$$116592\,{c_{\ell+4}}^2c_{\ell-10}{c_{\ell+1}}^2c_\ell + 3829\,c_{\ell+3}c_{\ell-7}c_{\ell+2}{c_{\ell+1}}^2c_\ell + 22064\,c_{\ell+4}c_{\ell-8}c_{\ell+2}{c_{\ell+1}}^2c_\ell +$$
$$416\,c_{\ell-6}{c_{\ell+2}}^2{c_{\ell+1}}^2c_\ell + 534\,c_{\ell+3}c_{\ell-6}{c_{\ell+1}}^3c_\ell + 3397\,c_{\ell+4}c_{\ell-7}{c_{\ell+1}}^3c_\ell + 11156\,{c_{\ell+3}}^3c_{\ell-9}{c_\ell}^2 +$$
$$122556\,c_{\ell+4}{c_{\ell+3}}^2c_{\ell-10}{c_\ell}^2 + 377632\,{c_{\ell+4}}^2c_{\ell+3}c_{\ell-11}{c_\ell}^2 + 341032\,{c_{\ell+4}}^3c_{\ell-12}{c_\ell}^2 +$$
$$12992\,{c_{\ell+3}}^2c_{\ell-8}c_{\ell+2}{c_\ell}^2 + 107590\,c_{\ell+4}c_{\ell+3}c_{\ell-9}c_{\ell+2}{c_\ell}^2 + 182046\,{c_{\ell+4}}^2c_{\ell-10}c_{\ell+2}{c_\ell}^2 +$$
$$4364\,c_{\ell+3}c_{\ell-7}{c_{\ell+2}}^2{c_\ell}^2 + 21214\,c_{\ell+4}c_{\ell-8}{c_{\ell+2}}^2{c_\ell}^2 + 398\,c_{\ell-6}{c_{\ell+2}}^3{c_\ell}^2 + 6804\,{c_{\ell+3}}^2c_{\ell-7}c_{\ell+1}{c_\ell}^2 +$$
$$59065\,c_{\ell+4}c_{\ell+3}c_{\ell-8}c_{\ell+1}{c_\ell}^2 + 104807\,{c_{\ell+4}}^2c_{\ell-9}c_{\ell+1}{c_\ell}^2 + 4138\,c_{\ell+3}c_{\ell-6}c_{\ell+2}c_{\ell+1}{c_\ell}^2 +$$
$$21441\,c_{\ell+4}c_{\ell-7}c_{\ell+2}c_{\ell+1}{c_\ell}^2 + 4939\,c_{\ell+4}c_{\ell-6}{c_{\ell+1}}^2{c_\ell}^2 + 2354\,{c_{\ell+3}}^2c_{\ell-6}{c_\ell}^3 +$$
$$18702\,c_{\ell+4}c_{\ell+3}c_{\ell-7}{c_\ell}^3 + 31422\,{c_{\ell+4}}^2c_{\ell-8}{c_\ell}^3 + 7004\,c_{\ell+4}c_{\ell-6}c_{\ell+2}{c_\ell}^3 + {c_\ell}^6 +$$
$$14860\,{c_{\ell+3}}^4c_{\ell-11}c_{\ell-1} + 203562\,c_{\ell+4}{c_{\ell+3}}^3c_{\ell-12}c_{\ell-1} + 883390\,{c_{\ell+4}}^2{c_{\ell+3}}^2c_{\ell-13}c_{\ell-1} +$$
$$1504976\,{c_{\ell+4}}^3c_{\ell+3}c_{\ell-14}c_{\ell-1} + 872404\,{c_{\ell+4}}^4c_{\ell-15}c_{\ell-1} + 26056\,{c_{\ell+3}}^3c_{\ell-10}c_{\ell+2}c_{\ell-1} +$$
$$296660\,c_{\ell+4}{c_{\ell+3}}^2c_{\ell-11}c_{\ell+2}c_{\ell-1} + 928718\,{c_{\ell+4}}^2c_{\ell+3}c_{\ell-12}c_{\ell+2}c_{\ell-1} + 843288\,{c_{\ell+4}}^3c_{\ell-13}c_{\ell+2}c_{\ell-1} +$$
$$15396\,{c_{\ell+3}}^2c_{\ell-9}{c_{\ell+2}}^2c_{\ell-1} + 132826\,c_{\ell+4}c_{\ell+3}c_{\ell-10}{c_{\ell+2}}^2c_{\ell-1} + 228588\,{c_{\ell+4}}^2c_{\ell-11}{c_{\ell+2}}^2c_{\ell-1} +$$
$$3512\,c_{\ell+3}c_{\ell-8}{c_{\ell+2}}^3c_{\ell-1} + 17904\,c_{\ell+4}c_{\ell-9}{c_{\ell+2}}^3c_{\ell-1} + 246\,c_{\ell-7}{c_{\ell+2}}^4c_{\ell-1} +$$
$$15634\,{c_{\ell+3}}^3c_{\ell-9}c_{\ell+1}c_{\ell-1} + 182336\,c_{\ell+4}{c_{\ell+3}}^2c_{\ell-10}c_{\ell+1}c_{\ell-1} + 587832\,{c_{\ell+4}}^2c_{\ell+3}c_{\ell-11}c_{\ell+1}c_{\ell-1} +$$
$$550100\,{c_{\ell+4}}^3c_{\ell-12}c_{\ell+1}c_{\ell-1} + 17451\,{c_{\ell+3}}^2c_{\ell-8}c_{\ell+2}c_{\ell+1}c_{\ell-1} + 155008\,c_{\ell+4}c_{\ell+3}c_{\ell-9}c_{\ell+2}c_{\ell+1}c_{\ell-1} +$$
$$276240\,{c_{\ell+4}}^2c_{\ell-10}c_{\ell+2}c_{\ell+1}c_{\ell-1} + 5565\,c_{\ell+3}c_{\ell-7}{c_{\ell+2}}^2c_{\ell+1}c_{\ell-1} + 29434\,c_{\ell+4}c_{\ell-8}{c_{\ell+2}}^2c_{\ell+1}c_{\ell-1} +$$
$$474\,c_{\ell-6}{c_{\ell+2}}^3c_{\ell+1}c_{\ell-1} + 4672\,{c_{\ell+3}}^2c_{\ell-7}{c_{\ell+1}}^2c_{\ell-1} + 43029\,c_{\ell+4}c_{\ell+3}c_{\ell-8}{c_{\ell+1}}^2c_{\ell-1} +$$
$$79863\,{c_{\ell+4}}^2c_{\ell-9}{c_{\ell+1}}^2c_{\ell-1} + 2737\,c_{\ell+3}c_{\ell-6}c_{\ell+2}{c_{\ell+1}}^2c_{\ell-1} + 15184\,c_{\ell+4}c_{\ell-7}c_{\ell+2}{c_{\ell+1}}^2c_{\ell-1} +$$
$$2424\,c_{\ell+4}c_{\ell-6}{c_{\ell+1}}^3c_{\ell-1} + 14898\,{c_{\ell+3}}^3c_{\ell-8}c_\ell c_{\ell-1} + 165332\,c_{\ell+4}{c_{\ell+3}}^2c_{\ell-9}c_\ell c_{\ell-1} +$$
$$516922\,{c_{\ell+4}}^2c_{\ell+3}c_{\ell-10}c_\ell c_{\ell-1} + 474948\,{c_{\ell+4}}^3c_{\ell-11}c_\ell c_{\ell-1} + 16948\,{c_{\ell+3}}^2c_{\ell-7}c_{\ell+2}c_\ell c_{\ell-1} +$$
$$141498\,c_{\ell+4}c_{\ell+3}c_{\ell-8}c_{\ell+2}c_\ell c_{\ell-1} + 242718\,{c_{\ell+4}}^2c_{\ell-9}c_{\ell+2}c_\ell c_{\ell-1} + 5578\,c_{\ell+3}c_{\ell-6}{c_{\ell+2}}^2c_\ell c_{\ell-1} +$$
$$27230\,c_{\ell+4}c_{\ell-7}{c_{\ell+2}}^2c_\ell c_{\ell-1} + 9079\,{c_{\ell+3}}^2c_{\ell-6}c_{\ell+1}c_\ell c_{\ell-1} + 78036\,c_{\ell+4}c_{\ell+3}c_{\ell-7}c_{\ell+1}c_\ell c_{\ell-1} +$$
$$138874\,{c_{\ell+4}}^2c_{\ell-8}c_{\ell+1}c_\ell c_{\ell-1} + 28028\,c_{\ell+4}c_{\ell-6}c_{\ell+2}c_{\ell+1}c_\ell c_{\ell-1} + 35043\,c_{\ell+4}c_{\ell+3}c_{\ell-6}{c_\ell}^2c_{\ell-1} +$$

$$59853\, {c_{\ell+4}}^2 c_{\ell-7} {c_{\ell}}^2 c_{\ell-1} + 15\, c_{\ell+1} {c_{\ell}}^4 c_{\ell-1} + 5006\, {c_{\ell+3}}^3 c_{\ell-7} {c_{\ell-1}}^2 + 55675\, c_{\ell+4} {c_{\ell+3}}^2 c_{\ell-8} {c_{\ell-1}}^2 +$$
$$175711\, {c_{\ell+4}}^2 c_{\ell+3} c_{\ell-9} {c_{\ell-1}}^2 + 163708\, {c_{\ell+4}}^3 c_{\ell-10} {c_{\ell-1}}^2 + 5695\, {c_{\ell+3}}^2 c_{\ell-6} c_{\ell+2} {c_{\ell-1}}^2 +$$
$$47095\, c_{\ell+4} c_{\ell+3} c_{\ell-7} c_{\ell+2} {c_{\ell-1}}^2 + 81008\, {c_{\ell+4}}^2 c_{\ell-8} c_{\ell+2} {c_{\ell-1}}^2 + 9072\, c_{\ell+4} c_{\ell-6} {c_{\ell+2}}^2 {c_{\ell-1}}^2 +$$
$$25739\, c_{\ell+4} c_{\ell+3} c_{\ell-6} c_{\ell+1} {c_{\ell-1}}^2 + 45762\, {c_{\ell+4}}^2 c_{\ell-7} c_{\ell+1} {c_{\ell-1}}^2 + 38104\, {c_{\ell+4}}^2 c_{\ell-6} c_{\ell} {c_{\ell-1}}^2 +$$
$$30\, {c_{\ell+1}}^2 {c_{\ell}}^2 {c_{\ell-1}}^2 + 55\, c_{\ell+2} {c_{\ell}}^3 {c_{\ell-1}}^2 + 5\, {c_{\ell+1}}^3 {c_{\ell-1}}^3 + 79\, c_{\ell+2} c_{\ell+1} c_{\ell} {c_{\ell-1}}^3 + 141\, c_{\ell+3} {c_{\ell}}^2 {c_{\ell-1}}^3 +$$
$$17\, {c_{\ell+2}}^2 {c_{\ell-1}}^4 + 55\, c_{\ell+3} c_{\ell+1} {c_{\ell-1}}^4 + 202\, c_{\ell+4} c_{\ell} {c_{\ell-1}}^4 + 12550\, {c_{\ell+3}}^4 c_{\ell-10} c_{\ell-2} +$$
$$168968\, c_{\ell+4} {c_{\ell+3}}^3 c_{\ell-11} c_{\ell-2} + 727416\, {c_{\ell+4}}^2 {c_{\ell+3}}^2 c_{\ell-12} c_{\ell-2} + 1236736\, {c_{\ell+4}}^3 c_{\ell+3} c_{\ell-13} c_{\ell-2} +$$
$$718216\, {c_{\ell+4}}^4 c_{\ell-14} c_{\ell-2} + 22858\, {c_{\ell+3}}^3 c_{\ell-9} c_{\ell+2} c_{\ell-2} + 251824\, c_{\ell+4} {c_{\ell+3}}^2 c_{\ell-10} c_{\ell+2} c_{\ell-2} +$$
$$773760\, {c_{\ell+4}}^2 c_{\ell+3} c_{\ell-11} c_{\ell+2} c_{\ell-2} + 696168\, {c_{\ell+4}}^3 c_{\ell-12} c_{\ell+2} c_{\ell-2} + 14360\, {c_{\ell+3}}^2 c_{\ell-8} {c_{\ell+2}}^2 c_{\ell-2} +$$
$$117200\, c_{\ell+4} c_{\ell+3} c_{\ell-9} {c_{\ell+2}}^2 c_{\ell-2} + 194828\, {c_{\ell+4}}^2 c_{\ell-10} {c_{\ell+2}}^2 c_{\ell-2} + 3596\, c_{\ell+3} c_{\ell-7} {c_{\ell+2}}^3 c_{\ell-2} +$$
$$16812\, c_{\ell+4} c_{\ell-8} {c_{\ell+2}}^3 c_{\ell-2} + 288\, c_{\ell-6} {c_{\ell+2}}^4 c_{\ell-2} + 12749\, {c_{\ell+3}}^3 c_{\ell-8} c_{\ell+1} c_{\ell-2} +$$
$$146488\, c_{\ell+4} {c_{\ell+3}}^2 c_{\ell-9} c_{\ell+1} c_{\ell-2} + 468924\, {c_{\ell+4}}^2 c_{\ell+3} c_{\ell-10} c_{\ell+1} c_{\ell-2} + 438260\, {c_{\ell+4}}^3 c_{\ell-11} c_{\ell+1} c_{\ell-2} +$$
$$15029\, {c_{\ell+3}}^2 c_{\ell-7} c_{\ell+2} c_{\ell+1} c_{\ell-2} + 128914\, c_{\ell+4} c_{\ell+3} c_{\ell-8} c_{\ell+2} c_{\ell+1} c_{\ell-2} +$$
$$224778\, {c_{\ell+4}}^2 c_{\ell-9} c_{\ell+2} c_{\ell+1} c_{\ell-2} + 5222\, c_{\ell+3} c_{\ell-6} {c_{\ell+2}}^2 c_{\ell+1} c_{\ell-2} + 25962\, c_{\ell+4} c_{\ell-7} {c_{\ell+2}}^2 c_{\ell+1} c_{\ell-2} +$$
$$3672\, {c_{\ell+3}}^2 c_{\ell-6} {c_{\ell+1}}^2 c_{\ell-2} + 33355\, c_{\ell+4} c_{\ell+3} c_{\ell-7} {c_{\ell+1}}^2 c_{\ell-2} + 61452\, {c_{\ell+4}}^2 c_{\ell-8} {c_{\ell+1}}^2 c_{\ell-2} +$$
$$12460\, c_{\ell+4} c_{\ell-6} c_{\ell+2} {c_{\ell+1}}^2 c_{\ell-2} + 12462\, {c_{\ell+3}}^3 c_{\ell-7} c_{\ell} c_{\ell-2} + 134758\, c_{\ell+4} {c_{\ell+3}}^2 c_{\ell-8} c_{\ell} c_{\ell-2} +$$
$$415486\, {c_{\ell+4}}^2 c_{\ell+3} c_{\ell-9} c_{\ell} c_{\ell-2} + 379516\, {c_{\ell+4}}^3 c_{\ell-10} c_{\ell} c_{\ell-2} + 15344\, {c_{\ell+3}}^2 c_{\ell-6} c_{\ell+2} c_{\ell} c_{\ell-2} +$$
$$121170\, c_{\ell+4} c_{\ell+3} c_{\ell-7} c_{\ell+2} c_{\ell} c_{\ell-2} + 200884\, {c_{\ell+4}}^2 c_{\ell-8} c_{\ell+2} c_{\ell} c_{\ell-2} + 25320\, c_{\ell+4} c_{\ell-6} {c_{\ell+2}}^2 c_{\ell} c_{\ell-2} +$$
$$61178\, c_{\ell+4} c_{\ell+3} c_{\ell-6} c_{\ell+1} c_{\ell} c_{\ell-2} + 107570\, {c_{\ell+4}}^2 c_{\ell-7} c_{\ell+1} c_{\ell} c_{\ell-2} + 47422\, {c_{\ell+4}}^2 c_{\ell-6} {c_{\ell}}^2 c_{\ell-2} +$$
$$20\, {c_{\ell+1}}^2 {c_{\ell}}^3 c_{\ell-2} + 50\, c_{\ell+2} {c_{\ell}}^4 c_{\ell-2} + 8885\, {c_{\ell+3}}^3 c_{\ell-6} c_{\ell-1} c_{\ell-2} + 93024\, c_{\ell+4} {c_{\ell+3}}^2 c_{\ell-7} c_{\ell-1} c_{\ell-2} +$$
$$283918\, {c_{\ell+4}}^2 c_{\ell+3} c_{\ell-8} c_{\ell-1} c_{\ell-2} + 260236\, {c_{\ell+4}}^3 c_{\ell-9} c_{\ell-1} c_{\ell-2} + 80766\, c_{\ell+4} c_{\ell+3} c_{\ell-6} c_{\ell+2} c_{\ell-1} c_{\ell-2} +$$
$$132742\, {c_{\ell+4}}^2 c_{\ell-7} c_{\ell+2} c_{\ell-1} c_{\ell-2} + 71562\, {c_{\ell+4}}^2 c_{\ell-6} c_{\ell+1} c_{\ell-1} c_{\ell-2} + 30\, {c_{\ell+1}}^3 c_{\ell} c_{\ell-1} c_{\ell-2} +$$
$$315\, c_{\ell+2} c_{\ell+1} {c_{\ell}}^2 c_{\ell-1} c_{\ell-2} + 425\, c_{\ell+3} {c_{\ell}}^3 c_{\ell-1} c_{\ell-2} + 109\, c_{\ell+2} {c_{\ell+1}}^2 {c_{\ell-1}}^2 c_{\ell-2} +$$
$$280\, {c_{\ell+2}}^2 c_{\ell} {c_{\ell-1}}^2 c_{\ell-2} + 861\, c_{\ell+3} c_{\ell+1} c_{\ell} {c_{\ell-1}}^2 c_{\ell-2} + 1704\, c_{\ell+4} {c_{\ell}}^2 {c_{\ell-1}}^2 c_{\ell-2} +$$
$$450\, c_{\ell+3} c_{\ell+2} {c_{\ell-1}}^3 c_{\ell-2} + 884\, c_{\ell+4} c_{\ell+1} {c_{\ell-1}}^3 c_{\ell-2} + 37419\, c_{\ell+4} {c_{\ell+3}}^2 c_{\ell-6} {c_{\ell-2}}^2 +$$
$$115387\, {c_{\ell+4}}^2 c_{\ell+3} c_{\ell-7} {c_{\ell-2}}^2 + 105914\, {c_{\ell+4}}^3 c_{\ell-8} {c_{\ell-2}}^2 + 56776\, {c_{\ell+4}}^2 c_{\ell-6} c_{\ell+2} {c_{\ell-2}}^2 + 3\, {c_{\ell+1}}^4 {c_{\ell-2}}^2 +$$
$$126\, c_{\ell+2} {c_{\ell+1}}^2 c_{\ell} {c_{\ell-2}}^2 + 247\, {c_{\ell+2}}^2 {c_{\ell}}^2 {c_{\ell-2}}^2 + 514\, c_{\ell+3} c_{\ell+1} {c_{\ell}}^2 {c_{\ell-2}}^2 + 818\, c_{\ell+4} {c_{\ell}}^3 {c_{\ell-2}}^2 +$$
$$269\, {c_{\ell+2}}^2 c_{\ell+1} c_{\ell-1} {c_{\ell-2}}^2 + 378\, c_{\ell+3} {c_{\ell+1}}^2 c_{\ell-1} {c_{\ell-2}}^2 + 1890\, c_{\ell+3} c_{\ell+2} c_{\ell} c_{\ell-1} {c_{\ell-2}}^2 +$$
$$3354\, c_{\ell+4} c_{\ell+1} c_{\ell} c_{\ell-1} {c_{\ell-2}}^2 + 919\, {c_{\ell+3}}^2 {c_{\ell-1}}^2 {c_{\ell-2}}^2 + 2594\, c_{\ell+4} c_{\ell+2} {c_{\ell-1}}^2 {c_{\ell-2}}^2 + 86\, {c_{\ell+2}}^3 {c_{\ell-2}}^3 +$$
$$544\, c_{\ell+3} c_{\ell+2} c_{\ell+1} {c_{\ell-2}}^3 + 520\, c_{\ell+4} {c_{\ell+1}}^2 {c_{\ell-2}}^3 + 706\, {c_{\ell+3}}^2 c_{\ell} {c_{\ell-2}}^3 + 2496\, c_{\ell+4} c_{\ell+2} c_{\ell} {c_{\ell-2}}^3 +$$
$$3868\, c_{\ell+4} c_{\ell+3} c_{\ell-1} {c_{\ell-2}}^3 + 1408\, {c_{\ell+4}}^2 {c_{\ell-2}}^4 + 13053\, {c_{\ell+3}}^4 c_{\ell-9} c_{\ell-3} + 162585\, c_{\ell+4} {c_{\ell+3}}^3 c_{\ell-10} c_{\ell-3} +$$

$$662518\, {c_{\ell+4}}^2 {c_{\ell+3}}^2 c_{\ell-11} c_{\ell-3} + 1086146\, {c_{\ell+4}}^3 c_{\ell+3} c_{\ell-12} c_{\ell-3} + 616729\, {c_{\ell+4}}^4 c_{\ell-13} c_{\ell-3} +$$

$$22592\, {c_{\ell+3}}^3 c_{\ell-8} c_{\ell+2} c_{\ell-3} + 231597\, c_{\ell+4} {c_{\ell+3}}^2 c_{\ell-9} c_{\ell+2} c_{\ell-3} + 675884\, {c_{\ell+4}}^2 c_{\ell+3} c_{\ell-10} c_{\ell+2} c_{\ell-3} +$$

$$588138\, {c_{\ell+4}}^3 c_{\ell-11} c_{\ell+2} c_{\ell-3} + 13198\, {c_{\ell+3}}^2 c_{\ell-7} {c_{\ell+2}}^2 c_{\ell-3} + 101198\, c_{\ell+4} c_{\ell+3} c_{\ell-8} {c_{\ell+2}}^2 c_{\ell-3} +$$

$$160751\, {c_{\ell+4}}^2 c_{\ell-9} {c_{\ell+2}}^2 c_{\ell-3} + 3008\, c_{\ell+3} c_{\ell-6} {c_{\ell+2}}^3 c_{\ell-3} + 13354\, c_{\ell+4} c_{\ell-7} {c_{\ell+2}}^3 c_{\ell-3} +$$

$$13177\, {c_{\ell+3}}^3 c_{\ell-7} c_{\ell+1} c_{\ell-3} + 138767\, c_{\ell+4} {c_{\ell+3}}^2 c_{\ell-8} c_{\ell+1} c_{\ell-3} + 416730\, {c_{\ell+4}}^2 c_{\ell+3} c_{\ell-9} c_{\ell+1} c_{\ell-3} +$$

$$373133\, {c_{\ell+4}}^3 c_{\ell-10} c_{\ell+1} c_{\ell-3} + 14664\, {c_{\ell+3}}^2 c_{\ell-6} c_{\ell+2} c_{\ell+1} c_{\ell-3} + 115675\, c_{\ell+4} c_{\ell+3} c_{\ell-7} c_{\ell+2} c_{\ell+1} c_{\ell-3} +$$

$$189526\, {c_{\ell+4}}^2 c_{\ell-8} c_{\ell+2} c_{\ell+1} c_{\ell-3} + 21868\, c_{\ell+4} c_{\ell-6} {c_{\ell+2}}^2 c_{\ell+1} c_{\ell-3} +$$

$$30570\, c_{\ell+4} c_{\ell+3} c_{\ell-6} {c_{\ell+1}}^2 c_{\ell-3} + 52521\, {c_{\ell+4}}^2 c_{\ell-7} {c_{\ell+1}}^2 c_{\ell-3} + 13660\, {c_{\ell+3}}^3 c_{\ell-6} c_{\ell} c_{\ell-3} +$$

$$131508\, c_{\ell+4} {c_{\ell+3}}^2 c_{\ell-7} c_{\ell} c_{\ell-3} + 373794\, {c_{\ell+4}}^2 c_{\ell+3} c_{\ell-8} c_{\ell} c_{\ell-3} + 323713\, {c_{\ell+4}}^3 c_{\ell-9} c_{\ell} c_{\ell-3} +$$

$$109200\, c_{\ell+4} c_{\ell+3} c_{\ell-6} c_{\ell+2} c_{\ell} c_{\ell-3} + 168266\, {c_{\ell+4}}^2 c_{\ell-7} c_{\ell+2} c_{\ell} c_{\ell-3} + 93414\, {c_{\ell+4}}^2 c_{\ell-6} c_{\ell+1} c_{\ell} c_{\ell-3} +$$

$$15\, {c_{\ell+1}}^3 {c_{\ell}}^2 c_{\ell-3} + 145\, c_{\ell+2} c_{\ell+1} {c_{\ell}}^3 c_{\ell-3} + 190\, c_{\ell+3} {c_{\ell}}^4 c_{\ell-3} + 88010\, c_{\ell+4} {c_{\ell+3}}^2 c_{\ell-6} c_{\ell-1} c_{\ell-3} +$$

$$257916\, {c_{\ell+4}}^2 c_{\ell+3} c_{\ell-7} c_{\ell-1} c_{\ell-3} + 227945\, {c_{\ell+4}}^3 c_{\ell-8} c_{\ell-1} c_{\ell-3} + 116326\, {c_{\ell+4}}^2 c_{\ell-6} c_{\ell+2} c_{\ell-1} c_{\ell-3} +$$

$$6\, {c_{\ell+1}}^4 c_{\ell-1} c_{\ell-3} + 273\, c_{\ell+2} {c_{\ell+1}}^2 c_{\ell} c_{\ell-1} c_{\ell-3} + 360\, {c_{\ell+2}}^2 {c_{\ell}}^2 c_{\ell-1} c_{\ell-3} + 1302\, c_{\ell+3} c_{\ell+1} {c_{\ell}}^2 c_{\ell-1} c_{\ell-3} +$$

$$1825\, c_{\ell+4} {c_{\ell}}^3 c_{\ell-1} c_{\ell-3} + 252\, {c_{\ell+2}}^2 c_{\ell+1} {c_{\ell-1}}^2 c_{\ell-3} + 451\, c_{\ell+3} {c_{\ell+1}}^2 {c_{\ell-1}}^2 c_{\ell-3} +$$

$$1863\, c_{\ell+3} c_{\ell+2} c_{\ell} {c_{\ell-1}}^2 c_{\ell-3} + 3920\, c_{\ell+4} c_{\ell+1} c_{\ell} {c_{\ell-1}}^2 c_{\ell-3} + 659\, {c_{\ell+3}}^2 {c_{\ell-1}}^3 c_{\ell-3} +$$

$$1815\, c_{\ell+4} c_{\ell+2} {c_{\ell-1}}^3 c_{\ell-3} + 201944\, {c_{\ell+4}}^2 c_{\ell+3} c_{\ell-6} c_{\ell-2} c_{\ell-3} + 174605\, {c_{\ell+4}}^3 c_{\ell-7} c_{\ell-2} c_{\ell-3} +$$

$$72\, c_{\ell+2} {c_{\ell+1}}^3 c_{\ell-2} c_{\ell-3} + 714\, {c_{\ell+2}}^2 c_{\ell+1} c_{\ell} c_{\ell-2} c_{\ell-3} + 988\, c_{\ell+3} {c_{\ell+1}}^2 c_{\ell} c_{\ell-2} c_{\ell-3} +$$

$$2835\, c_{\ell+3} c_{\ell+2} {c_{\ell}}^2 c_{\ell-2} c_{\ell-3} + 4787\, c_{\ell+4} c_{\ell+1} {c_{\ell}}^2 c_{\ell-2} c_{\ell-3} + 376\, {c_{\ell+2}}^3 c_{\ell-1} c_{\ell-2} c_{\ell-3} +$$

$$3584\, c_{\ell+3} c_{\ell+2} c_{\ell+1} c_{\ell-1} c_{\ell-2} c_{\ell-3} + 3570\, c_{\ell+4} {c_{\ell+1}}^2 c_{\ell-1} c_{\ell-2} c_{\ell-3} + 5487\, {c_{\ell+3}}^2 c_{\ell} c_{\ell-1} c_{\ell-2} c_{\ell-3} +$$

$$14532\, c_{\ell+4} c_{\ell+2} c_{\ell} c_{\ell-1} c_{\ell-2} c_{\ell-3} + 13119\, c_{\ell+4} c_{\ell+3} {c_{\ell-1}}^2 c_{\ell-2} c_{\ell-3} + 1907\, c_{\ell+3} {c_{\ell+2}}^2 {c_{\ell-2}}^2 c_{\ell-3} +$$

$$2243\, {c_{\ell+3}}^2 c_{\ell+1} {c_{\ell-2}}^2 c_{\ell-3} + 6973\, c_{\ell+4} c_{\ell+2} c_{\ell+1} {c_{\ell-2}}^2 c_{\ell-3} + 16395\, c_{\ell+4} c_{\ell+3} c_{\ell} {c_{\ell-2}}^2 c_{\ell-3} +$$

$$17198\, {c_{\ell+4}}^2 c_{\ell-1} {c_{\ell-2}}^2 c_{\ell-3} + 75298\, {c_{\ell+4}}^3 c_{\ell-6} {c_{\ell-3}}^2 + 149\, {c_{\ell+2}}^2 {c_{\ell+1}}^2 {c_{\ell-3}}^2 + 148\, c_{\ell+3} {c_{\ell+1}}^3 {c_{\ell-3}}^2 +$$

$$230\, {c_{\ell+2}}^3 c_{\ell} {c_{\ell-3}}^2 + 2498\, c_{\ell+3} c_{\ell+2} c_{\ell+1} c_{\ell} {c_{\ell-3}}^2 + 2314\, c_{\ell+4} {c_{\ell+1}}^2 c_{\ell} {c_{\ell-3}}^2 + 2300\, {c_{\ell+3}}^2 {c_{\ell}}^2 {c_{\ell-3}}^2 +$$

$$4984\, c_{\ell+4} c_{\ell+2} {c_{\ell}}^2 {c_{\ell-3}}^2 + 1711\, c_{\ell+3} {c_{\ell+2}}^2 c_{\ell-1} {c_{\ell-3}}^2 + 2829\, {c_{\ell+3}}^2 c_{\ell+1} c_{\ell-1} {c_{\ell-3}}^2 +$$

$$7280\, c_{\ell+4} c_{\ell+2} c_{\ell+1} c_{\ell-1} {c_{\ell-3}}^2 + 19326\, c_{\ell+4} c_{\ell+3} c_{\ell} c_{\ell-1} {c_{\ell-3}}^2 + 9907\, {c_{\ell+4}}^2 {c_{\ell-1}}^2 {c_{\ell-3}}^2 +$$

$$5006\, {c_{\ell+3}}^2 c_{\ell+2} c_{\ell-2} {c_{\ell-3}}^2 + 6534\, c_{\ell+4} {c_{\ell+2}}^2 c_{\ell-2} {c_{\ell-3}}^2 + 16792\, c_{\ell+4} c_{\ell+3} c_{\ell+1} c_{\ell-2} {c_{\ell-3}}^2 +$$

$$22684\, {c_{\ell+4}}^2 c_{\ell} c_{\ell-2} {c_{\ell-3}}^2 + 1555\, {c_{\ell+3}}^3 {c_{\ell-3}}^3 + 10356\, c_{\ell+4} c_{\ell+3} c_{\ell+2} {c_{\ell-3}}^3 + 7701\, {c_{\ell+4}}^2 c_{\ell+1} {c_{\ell-3}}^3 +$$

$$10604\, {c_{\ell+3}}^4 c_{\ell-8} c_{\ell-4} + 139730\, c_{\ell+4} {c_{\ell+3}}^3 c_{\ell-9} c_{\ell-4} + 597684\, {c_{\ell+4}}^2 {c_{\ell+3}}^2 c_{\ell-10} c_{\ell-4} +$$

$$1014844\, {c_{\ell+4}}^3 c_{\ell+3} c_{\ell-11} c_{\ell-4} + 589618\, {c_{\ell+4}}^4 c_{\ell-12} c_{\ell-4} + 18916\, {c_{\ell+3}}^3 c_{\ell-7} c_{\ell+2} c_{\ell-4} +$$

$$204042\, c_{\ell+4} {c_{\ell+3}}^2 c_{\ell-8} c_{\ell+2} c_{\ell-4} + 625092\, {c_{\ell+4}}^2 c_{\ell+3} c_{\ell-9} c_{\ell+2} c_{\ell-4} + 563624\, {c_{\ell+4}}^3 c_{\ell-10} c_{\ell+2} c_{\ell-4} +$$

$$11778\, {c_{\ell+3}}^2 c_{\ell-6} {c_{\ell+2}}^2 c_{\ell-4} + 93304\, c_{\ell+4} c_{\ell+3} c_{\ell-7} {c_{\ell+2}}^2 c_{\ell-4} + 154718\, {c_{\ell+4}}^2 c_{\ell-8} {c_{\ell+2}}^2 c_{\ell-4} +$$

$$13332\, c_{\ell+4} c_{\ell-6} {c_{\ell+2}}^3 c_{\ell-4} + 11818\, {c_{\ell+3}}^3 c_{\ell-6} c_{\ell+1} c_{\ell-4} + 126378\, c_{\ell+4} {c_{\ell+3}}^2 c_{\ell-7} c_{\ell+1} c_{\ell-4} +$$
$$391910\, {c_{\ell+4}}^2 c_{\ell+3} c_{\ell-8} c_{\ell+1} c_{\ell-4} + 361296\, {c_{\ell+4}}^3 c_{\ell-9} c_{\ell+1} c_{\ell-4} +$$
$$107128\, c_{\ell+4} c_{\ell+3} c_{\ell-6} c_{\ell+2} c_{\ell+1} c_{\ell-4} + 181964\, {c_{\ell+4}}^2 c_{\ell-7} c_{\ell+2} c_{\ell+1} c_{\ell-4} +$$
$$51716\, {c_{\ell+4}}^2 c_{\ell-6} {c_{\ell+1}}^2 c_{\ell-4} + 116368\, c_{\ell+4} {c_{\ell+3}}^2 c_{\ell-6} c_{\ell} c_{\ell-4} + 356384\, {c_{\ell+4}}^2 c_{\ell+3} c_{\ell-7} c_{\ell} c_{\ell-4} +$$
$$323698\, {c_{\ell+4}}^3 c_{\ell-8} c_{\ell} c_{\ell-4} + 169912\, {c_{\ell+4}}^2 c_{\ell-6} c_{\ell+2} c_{\ell} c_{\ell-4} + 6\, {c_{\ell+1}}^4 c_{\ell} c_{\ell-4} +$$
$$171\, c_{\ell+2} {c_{\ell+1}}^2 {c_{\ell}}^2 c_{\ell-4} + 202\, {c_{\ell+2}}^2 {c_{\ell}}^3 c_{\ell-4} + 649\, c_{\ell+3} c_{\ell+1} {c_{\ell}}^3 c_{\ell-4} + 806\, c_{\ell+4} {c_{\ell}}^4 c_{\ell-4} +$$
$$237874\, {c_{\ell+4}}^2 c_{\ell+3} c_{\ell-6} c_{\ell-1} c_{\ell-4} + 216340\, {c_{\ell+4}}^3 c_{\ell-7} c_{\ell-1} c_{\ell-4} + 79\, c_{\ell+2} {c_{\ell+1}}^3 c_{\ell-1} c_{\ell-4} +$$
$$680\, {c_{\ell+2}}^2 c_{\ell+1} c_{\ell} c_{\ell-1} c_{\ell-4} + 1267\, c_{\ell+3} {c_{\ell+1}}^2 c_{\ell} c_{\ell-1} c_{\ell-4} + 2757\, c_{\ell+3} c_{\ell+2} {c_{\ell}}^2 c_{\ell-1} c_{\ell-4} +$$
$$6146\, c_{\ell+4} c_{\ell+1} {c_{\ell}}^2 c_{\ell-1} c_{\ell-4} + 184\, {c_{\ell+2}}^3 {c_{\ell-1}}^2 c_{\ell-4} + 1911\, c_{\ell+3} c_{\ell+2} c_{\ell+1} {c_{\ell-1}}^2 c_{\ell-4} +$$
$$2258\, c_{\ell+4} {c_{\ell+1}}^2 {c_{\ell-1}}^2 c_{\ell-4} + 2790\, {c_{\ell+3}}^2 c_{\ell} {c_{\ell-1}}^2 c_{\ell-4} + 8218\, c_{\ell+4} c_{\ell+2} c_{\ell} {c_{\ell-1}}^2 c_{\ell-4} +$$
$$4888\, c_{\ell+4} c_{\ell+3} {c_{\ell-1}}^3 c_{\ell-4} + 174094\, {c_{\ell+4}}^3 c_{\ell-6} c_{\ell-2} c_{\ell-4} + 298\, {c_{\ell+2}}^2 {c_{\ell+1}}^2 c_{\ell-2} c_{\ell-4} +$$
$$324\, c_{\ell+3} {c_{\ell+1}}^3 c_{\ell-2} c_{\ell-4} + 620\, {c_{\ell+2}}^3 c_{\ell} c_{\ell-2} c_{\ell-4} + 4956\, c_{\ell+3} c_{\ell+2} c_{\ell+1} c_{\ell} c_{\ell-2} c_{\ell-4} +$$
$$5110\, c_{\ell+4} {c_{\ell+1}}^2 c_{\ell} c_{\ell-2} c_{\ell-4} + 3717\, {c_{\ell+3}}^2 {c_{\ell}}^2 c_{\ell-2} c_{\ell-4} + 11712\, c_{\ell+4} c_{\ell+2} {c_{\ell}}^2 c_{\ell-2} c_{\ell-4} +$$
$$3718\, c_{\ell+3} {c_{\ell+2}}^2 c_{\ell-1} c_{\ell-2} c_{\ell-4} + 5299\, {c_{\ell+3}}^2 c_{\ell+1} c_{\ell-1} c_{\ell-2} c_{\ell-4} + 15846\, c_{\ell+4} c_{\ell+2} c_{\ell+1} c_{\ell-1} c_{\ell-2} c_{\ell-4} +$$
$$38304\, c_{\ell+4} c_{\ell+3} c_{\ell} c_{\ell-1} c_{\ell-2} c_{\ell-4} + 20780\, {c_{\ell+4}}^2 {c_{\ell-1}}^2 c_{\ell-2} c_{\ell-4} + 4546\, {c_{\ell+3}}^2 c_{\ell+2} {c_{\ell-2}}^2 c_{\ell-4} +$$
$$7760\, c_{\ell+4} {c_{\ell+2}}^2 {c_{\ell-2}}^2 c_{\ell-4} + 17312\, c_{\ell+4} c_{\ell+3} c_{\ell+1} {c_{\ell-2}}^2 c_{\ell-4} + 26886\, {c_{\ell+4}}^2 c_{\ell} {c_{\ell-2}}^2 c_{\ell-4} +$$
$$452\, {c_{\ell+2}}^3 c_{\ell+1} c_{\ell-3} c_{\ell-4} + 2248\, c_{\ell+3} c_{\ell+2} {c_{\ell+1}}^2 c_{\ell-3} c_{\ell-4} + 1598\, c_{\ell+4} {c_{\ell+1}}^3 c_{\ell-3} c_{\ell-4} +$$
$$4992\, c_{\ell+3} {c_{\ell+2}}^2 c_{\ell} c_{\ell-3} c_{\ell-4} + 7916\, {c_{\ell+3}}^2 c_{\ell+1} c_{\ell} c_{\ell-3} c_{\ell-4} + 20874\, c_{\ell+4} c_{\ell+2} c_{\ell+1} c_{\ell} c_{\ell-3} c_{\ell-4} +$$
$$28924\, c_{\ell+4} c_{\ell+3} {c_{\ell}}^2 c_{\ell-3} c_{\ell-4} + 9890\, {c_{\ell+3}}^2 c_{\ell+2} c_{\ell-1} c_{\ell-3} c_{\ell-4} + 14106\, c_{\ell+4} {c_{\ell+2}}^2 c_{\ell-1} c_{\ell-3} c_{\ell-4} +$$
$$40824\, c_{\ell+4} c_{\ell+3} c_{\ell+1} c_{\ell-1} c_{\ell-3} c_{\ell-4} + 59472\, {c_{\ell+4}}^2 c_{\ell} c_{\ell-1} c_{\ell-3} c_{\ell-4} + 7584\, {c_{\ell+3}}^3 c_{\ell-2} c_{\ell-3} c_{\ell-4} +$$
$$64776\, c_{\ell+4} c_{\ell+3} c_{\ell+2} c_{\ell-2} c_{\ell-3} c_{\ell-4} + 52844\, {c_{\ell+4}}^2 c_{\ell+1} c_{\ell-2} c_{\ell-3} c_{\ell-4} + 36856\, c_{\ell+4} {c_{\ell+3}}^2 {c_{\ell-3}}^2 c_{\ell-4} +$$
$$43620\, {c_{\ell+4}}^2 c_{\ell+2} {c_{\ell-3}}^2 c_{\ell-4} + 134\, {c_{\ell+2}}^4 {c_{\ell-4}}^2 + 2274\, c_{\ell+3} {c_{\ell+2}}^2 c_{\ell+1} {c_{\ell-4}}^2 +$$
$$1878\, {c_{\ell+3}}^2 {c_{\ell+1}}^2 {c_{\ell-4}}^2 + 5177\, c_{\ell+4} c_{\ell+2} {c_{\ell+1}}^2 {c_{\ell-4}}^2 + 6570\, {c_{\ell+3}}^2 c_{\ell+2} c_{\ell} {c_{\ell-4}}^2 +$$
$$10590\, c_{\ell+4} {c_{\ell+2}}^2 c_{\ell} {c_{\ell-4}}^2 + 29197\, c_{\ell+4} c_{\ell+3} c_{\ell+1} c_{\ell} {c_{\ell-4}}^2 + 23858\, {c_{\ell+4}}^2 {c_{\ell}}^2 {c_{\ell-4}}^2 +$$
$$3830\, {c_{\ell+3}}^3 c_{\ell-1} {c_{\ell-4}}^2 + 35183\, c_{\ell+4} c_{\ell+3} c_{\ell+2} c_{\ell-1} {c_{\ell-4}}^2 + 33638\, {c_{\ell+4}}^2 c_{\ell+1} c_{\ell-1} {c_{\ell-4}}^2 +$$
$$34457\, c_{\ell+4} {c_{\ell+3}}^2 c_{\ell-2} {c_{\ell-4}}^2 + 50628\, {c_{\ell+4}}^2 c_{\ell+2} c_{\ell-2} {c_{\ell-4}}^2 + 97940\, {c_{\ell+4}}^2 c_{\ell+3} c_{\ell-3} {c_{\ell-4}}^2 +$$
$$29150\, {c_{\ell+4}}^3 {c_{\ell-4}}^3 + 10065\, {c_{\ell+3}}^4 c_{\ell-7} c_{\ell-5} + 127052\, c_{\ell+4} {c_{\ell+3}}^3 c_{\ell-8} c_{\ell-5} +$$
$$525675\, {c_{\ell+4}}^2 {c_{\ell+3}}^2 c_{\ell-9} c_{\ell-5} + 871344\, {c_{\ell+4}}^3 c_{\ell+3} c_{\ell-10} c_{\ell-5} + 498121\, {c_{\ell+4}}^4 c_{\ell-11} c_{\ell-5} +$$
$$19518\, {c_{\ell+3}}^3 c_{\ell-6} c_{\ell+2} c_{\ell-5} + 194145\, c_{\ell+4} {c_{\ell+3}}^2 c_{\ell-7} c_{\ell+2} c_{\ell-5} + 561576\, {c_{\ell+4}}^2 c_{\ell+3} c_{\ell-8} c_{\ell+2} c_{\ell-5} +$$
$$486981\, {c_{\ell+4}}^3 c_{\ell-9} c_{\ell+2} c_{\ell-5} + 89810\, c_{\ell+4} c_{\ell+3} c_{\ell-6} {c_{\ell+2}}^2 c_{\ell-5} + 138463\, {c_{\ell+4}}^2 c_{\ell-7} {c_{\ell+2}}^2 c_{\ell-5} +$$
$$116173\, c_{\ell+4} {c_{\ell+3}}^2 c_{\ell-6} c_{\ell+1} c_{\ell-5} + 357294\, {c_{\ell+4}}^2 c_{\ell+3} c_{\ell-7} c_{\ell+1} c_{\ell-5} + 323215\, {c_{\ell+4}}^3 c_{\ell-8} c_{\ell+1} c_{\ell-5} +$$

$$172634\, c_{\ell+4}{}^2 c_{\ell-6} c_{\ell+2} c_{\ell+1} c_{\ell-5} + c_{\ell+1}{}^5 c_{\ell-5} + 313200\, c_{\ell+4}{}^2 c_{\ell+3} c_{\ell-6} c_\ell c_{\ell-5} +$$
$$273327\, c_{\ell+4}{}^3 c_{\ell-7} c_\ell c_{\ell-5} + 94\, c_{\ell+2} c_{\ell+1}{}^3 c_\ell c_{\ell-5} + 493\, c_{\ell+2}{}^2 c_{\ell+1} c_\ell{}^2 c_{\ell-5} + 871\, c_{\ell+3} c_{\ell+1}{}^2 c_\ell{}^2 c_{\ell-5} +$$
$$1524\, c_{\ell+3} c_{\ell+2} c_\ell{}^3 c_{\ell-5} + 3225\, c_{\ell+4} c_{\ell+1} c_\ell{}^3 c_{\ell-5} + 191091\, c_{\ell+4}{}^3 c_{\ell-6} c_{\ell-1} c_{\ell-5} +$$
$$311\, c_{\ell+2}{}^2 c_{\ell+1}{}^2 c_{\ell-1} c_{\ell-5} + 409\, c_{\ell+3} c_{\ell+1}{}^3 c_{\ell-1} c_{\ell-5} + 502\, c_{\ell+2}{}^3 c_\ell c_{\ell-1} c_{\ell-5} +$$
$$5544\, c_{\ell+3} c_{\ell+2} c_{\ell+1} c_\ell c_{\ell-1} c_{\ell-5} + 6719\, c_{\ell+4} c_{\ell+1}{}^2 c_\ell c_{\ell-1} c_{\ell-5} + 4361\, c_{\ell+3}{}^2 c_\ell{}^2 c_{\ell-1} c_{\ell-5} +$$
$$12761\, c_{\ell+4} c_{\ell+2} c_\ell{}^2 c_{\ell-1} c_{\ell-5} + 1918\, c_{\ell+3} c_{\ell+2}{}^2 c_{\ell-1}{}^2 c_{\ell-5} + 3034\, c_{\ell+3}{}^2 c_{\ell+1} c_{\ell-1}{}^2 c_{\ell-5} +$$
$$9318\, c_{\ell+4} c_{\ell+2} c_{\ell+1} c_{\ell-1}{}^2 c_{\ell-5} + 22225\, c_{\ell+4} c_{\ell+3} c_\ell c_{\ell-1}{}^2 c_{\ell-5} + 8170\, c_{\ell+4}{}^2 c_{\ell-1}{}^3 c_{\ell-5} +$$
$$502\, c_{\ell+2}{}^3 c_{\ell+1} c_{\ell-2} c_{\ell-5} + 2337\, c_{\ell+3} c_{\ell+2} c_{\ell+1}{}^2 c_{\ell-2} c_{\ell-5} + 1851\, c_{\ell+4} c_{\ell+1}{}^3 c_{\ell-2} c_{\ell-5} +$$
$$5726\, c_{\ell+3} c_{\ell+2}{}^2 c_\ell c_{\ell-2} c_{\ell-5} + 7253\, c_{\ell+3}{}^2 c_{\ell+1} c_\ell c_{\ell-2} c_{\ell-5} + 23716\, c_{\ell+4} c_{\ell+2} c_{\ell+1} c_\ell c_{\ell-2} c_{\ell-5} +$$
$$28552\, c_{\ell+4} c_{\ell+3} c_\ell{}^2 c_{\ell-2} c_{\ell-5} + 10535\, c_{\ell+3}{}^2 c_{\ell+2} c_{\ell-1} c_{\ell-2} c_{\ell-5} + 16078\, c_{\ell+4} c_{\ell+2}{}^2 c_{\ell-1} c_{\ell-2} c_{\ell-5} +$$
$$41944\, c_{\ell+4} c_{\ell+3} c_{\ell+1} c_{\ell-1} c_{\ell-2} c_{\ell-5} + 61118\, c_{\ell+4}{}^2 c_\ell c_{\ell-1} c_{\ell-2} c_{\ell-5} + 3487\, c_{\ell+3}{}^3 c_{\ell-2}{}^2 c_{\ell-5} +$$
$$34763\, c_{\ell+4} c_{\ell+3} c_{\ell+2} c_{\ell-2}{}^2 c_{\ell-5} + 29392\, c_{\ell+4}{}^2 c_{\ell+1} c_{\ell-2}{}^2 c_{\ell-5} + 218\, c_{\ell+2}{}^4 c_{\ell-3} c_{\ell-5} +$$
$$4803\, c_{\ell+3} c_{\ell+2}{}^2 c_{\ell+1} c_{\ell-3} c_{\ell-5} + 3664\, c_{\ell+3}{}^2 c_{\ell+1}{}^2 c_{\ell-3} c_{\ell-5} + 10891\, c_{\ell+4} c_{\ell+2} c_{\ell+1}{}^2 c_{\ell-3} c_{\ell-5} +$$
$$15322\, c_{\ell+3}{}^2 c_{\ell+2} c_\ell c_{\ell-3} c_{\ell-5} + 20328\, c_{\ell+4} c_{\ell+2}{}^2 c_\ell c_{\ell-3} c_{\ell-5} + 58543\, c_{\ell+4} c_{\ell+3} c_{\ell+1} c_\ell c_{\ell-3} c_{\ell-5} +$$
$$41968\, c_{\ell+4}{}^2 c_\ell{}^2 c_{\ell-3} c_{\ell-5} + 8767\, c_{\ell+3}{}^3 c_{\ell-1} c_{\ell-3} c_{\ell-5} + 73255\, c_{\ell+4} c_{\ell+3} c_{\ell+2} c_{\ell-1} c_{\ell-3} c_{\ell-5} +$$
$$65506\, c_{\ell+4}{}^2 c_{\ell+1} c_{\ell-1} c_{\ell-3} c_{\ell-5} + 71679\, c_{\ell+4} c_{\ell+3}{}^2 c_{\ell-2} c_{\ell-3} c_{\ell-5} + 94662\, c_{\ell+4}{}^2 c_{\ell+2} c_{\ell-2} c_{\ell-3} c_{\ell-5} +$$
$$93096\, c_{\ell+4}{}^2 c_{\ell+3} c_{\ell-3}{}^2 c_{\ell-5} + 3028\, c_{\ell+3} c_{\ell+2}{}^3 c_{\ell-4} c_{\ell-5} + 13434\, c_{\ell+3}{}^2 c_{\ell+2} c_{\ell+1} c_{\ell-4} c_{\ell-5} +$$
$$20700\, c_{\ell+4} c_{\ell+2}{}^2 c_{\ell+1} c_{\ell-4} c_{\ell-5} + 29910\, c_{\ell+4} c_{\ell+3} c_{\ell+1}{}^2 c_{\ell-4} c_{\ell-5} + 10964\, c_{\ell+3}{}^3 c_\ell c_{\ell-4} c_{\ell-5} +$$
$$102284\, c_{\ell+4} c_{\ell+3} c_{\ell+2} c_\ell c_{\ell-4} c_{\ell-5} + 97076\, c_{\ell+4}{}^2 c_{\ell+1} c_\ell c_{\ell-4} c_{\ell-5} + 78722\, c_{\ell+4} c_{\ell+3}{}^2 c_{\ell-1} c_{\ell-4} c_{\ell-5} +$$
$$112104\, c_{\ell+4}{}^2 c_{\ell+2} c_{\ell-1} c_{\ell-4} c_{\ell-5} + 196512\, c_{\ell+4}{}^2 c_{\ell+3} c_{\ell-2} c_{\ell-4} c_{\ell-5} + 155342\, c_{\ell+4}{}^3 c_{\ell-3} c_{\ell-4} c_{\ell-5} +$$
$$6288\, c_{\ell+3}{}^2 c_{\ell+2}{}^2 c_{\ell-5}{}^2 + 6026\, c_{\ell+4} c_{\ell+2}{}^3 c_{\ell-5}{}^2 + 5356\, c_{\ell+3}{}^3 c_{\ell+1} c_{\ell-5}{}^2 +$$
$$52387\, c_{\ell+4} c_{\ell+3} c_{\ell+2} c_{\ell+1} c_{\ell-5}{}^2 + 26170\, c_{\ell+4}{}^2 c_{\ell+1}{}^2 c_{\ell-5}{}^2 + 54086\, c_{\ell+4} c_{\ell+3}{}^2 c_\ell c_{\ell-5}{}^2 +$$
$$76748\, c_{\ell+4}{}^2 c_{\ell+2} c_\ell c_{\ell-5}{}^2 + 110166\, c_{\ell+4}{}^2 c_{\ell+3} c_{\ell-1} c_{\ell-5}{}^2 + 79459\, c_{\ell+4}{}^3 c_{\ell-2} c_{\ell-5}{}^2$$

The multiple point loci for our special case were called y_n. These are the numbers that match exactly with the Euler characteristics extracted from (2.8).

$$y_1 = \frac{1}{2}L^2 - \frac{1}{2}K.L + 1$$

$$y_2 = -\frac{1}{24}r^4K^2 + \frac{1}{4}r^4 - \frac{1}{6}r^3K.L + \frac{1}{12}r^3K^2 - \frac{1}{2}r^2L^2 + \frac{1}{2}r^2K.L + \frac{1}{24}r^2K^2 - \frac{5}{4}r^2$$
$$+ \frac{1}{8}(L^2)^2 + \frac{1}{6}rK.L - \frac{1}{4}(L^2)(K.L) + \frac{1}{8}(K.L)^2 - \frac{1}{12}rK^2 + \frac{3}{4}L^2 - \frac{3}{4}K.L + 1$$

From here, it becomes convenient to compactify the notation by setting

$$
\begin{aligned}
k &= K^2 \\
\ell &= L^2 \\
j &= K.L
\end{aligned}
$$

$$
\begin{aligned}
y_3 = {} & \frac{97}{720}r^6k - \frac{2}{3}r^6 + \frac{17}{40}r^5j - \frac{17}{80}r^5k - \frac{1}{48}r^4\ell k + \frac{1}{48}r^4jk + \frac{7}{8}r^4\ell - \frac{7}{8}r^4j - \frac{1}{12}r^3\ell j + \frac{1}{12}r^3j^2 \\
& - \frac{37}{144}r^4k + \frac{1}{24}r^3\ell k - \frac{1}{24}r^3jk + \frac{11}{4}r^4 - \frac{1}{4}r^2\ell^2 - \frac{19}{24}r^3j + \frac{1}{2}r^2\ell j - \frac{1}{4}r^2j^2 + \frac{19}{48}r^3k \\
& + \frac{1}{48}r^2\ell k - \frac{1}{48}r^2jk - \frac{15}{8}r^2\ell + \frac{1}{48}\ell^3 + \frac{15}{8}r^2j + \frac{1}{12}r\ell j - \frac{1}{16}\ell^2j - \frac{1}{12}rj^2 + \frac{1}{16}\ell j^2 - \frac{1}{48}j^3 \\
& + \frac{11}{90}r^2k - \frac{1}{24}r\ell k + \frac{1}{24}rjk - \frac{37}{12}r^2 + \frac{1}{4}\ell^2 + \frac{11}{30}rj - \frac{1}{2}\ell j + \frac{1}{4}j^2 - \frac{11}{60}rk + \frac{11}{12}\ell - \frac{11}{12}j + 1
\end{aligned}
$$

$y_4 = \frac{1}{1152}r^8k^2 - \frac{853}{2240}r^8k + \frac{1}{144}r^7jk - \frac{1}{288}r^7k^2 + \frac{53}{32}r^8 - \frac{2629}{2520}r^7j + \frac{1}{72}r^6j^2 + \frac{2629}{5040}r^7k + \frac{127}{1440}r^6\ell k - \frac{49}{480}r^6jk + \frac{1}{576}r^6k^2 - \frac{43}{24}r^6\ell + \frac{43}{24}r^6j + \frac{71}{240}r^5\ell j - \frac{71}{240}r^5j^2 + \frac{1423}{1440}r^6k - \frac{71}{480}r^5\ell k - \frac{1}{192}r^4\ell^2k + \frac{193}{1440}r^5jk + \frac{1}{96}r^4\ell jk - \frac{1}{192}r^4j^2k + \frac{1}{144}r^5k^2 - \frac{331}{48}r^6 + \frac{17}{32}r^4\ell^2 + \frac{947}{360}r^5j - \frac{17}{16}r^4\ell j - \frac{1}{48}r^3\ell^2j + \frac{145}{288}r^4j^2 + \frac{1}{24}r^3\ell j^2 - \frac{1}{48}r^3j^3 - \frac{947}{720}r^5k - \frac{23}{144}r^4\ell k + \frac{1}{96}r^3\ell^2k + \frac{3}{16}r^4jk - \frac{1}{48}r^3\ell jk + \frac{1}{96}r^3j^2k - \frac{7}{1152}r^4k^2 + \frac{77}{16}r^4\ell - \frac{1}{16}r^2\ell^3 - \frac{77}{16}r^4j - \frac{25}{48}r^3\ell j + \frac{3}{16}r^2\ell^2j + \frac{25}{48}r^3j^2 - \frac{3}{16}r^2\ell j^2 + \frac{1}{16}r^2j^3 - \frac{2419}{2880}r^4k + \frac{25}{96}r^3\ell k + \frac{1}{192}r^2\ell^2k - \frac{73}{288}r^3jk - \frac{1}{96}r^2\ell jk + \frac{1}{192}r^2j^2k - \frac{1}{288}r^3k^2 + \frac{305}{32}r^4 - \frac{29}{32}r^2\ell^2 + \frac{1}{384}\ell^4 - \frac{389}{180}r^3j + \frac{29}{16}r^2\ell j + \frac{1}{48}r\ell^2j - \frac{1}{96}\ell^3j - \frac{257}{288}r^2j^2 - \frac{1}{24}r\ell j^2 + \frac{1}{64}\ell^2j^2 + \frac{1}{48}rj^3 - \frac{1}{96}\ell j^3 + \frac{1}{384}j^4 + \frac{389}{360}r^3k + \frac{103}{1440}r^2\ell k - \frac{1}{96}r\ell^2k - \frac{41}{480}r^2jk + \frac{1}{48}r\ell jk - \frac{1}{96}rj^2k + \frac{1}{288}r^2k^2 - \frac{193}{48}r^2\ell + \frac{5}{96}\ell^3 + \frac{193}{48}r^2j + \frac{9}{40}r\ell j - \frac{5}{32}\ell^2j - \frac{9}{40}rj^2 + \frac{5}{32}\ell j^2 - \frac{5}{96}j^3 + \frac{293}{1260}r^2k - \frac{9}{80}r\ell k + \frac{9}{80}rjk - \frac{127}{24}r^2 + \frac{35}{96}\ell^2 + \frac{241}{420}rj - \frac{35}{48}\ell j + \frac{35}{96}j^2 - \frac{241}{840}rk + \frac{25}{24}\ell - \frac{25}{24}j + 1$

$y_5 = -\frac{97}{17280}r^{10}k^2 + \frac{538147}{518400}r^{10}k - \frac{347}{8640}r^9jk + \frac{347}{17280}r^9k^2 + \frac{1}{2304}r^8\ell k^2 - \frac{1}{2304}r^8jk^2 - \frac{41}{10}r^{10} + \frac{934951}{362880}r^9j - \frac{17}{240}r^8j^2 - \frac{934951}{725760}r^9k - \frac{11653}{40320}r^8\ell k + \frac{14509}{40320}r^8jk + \frac{1}{288}r^7\ell jk - \frac{1}{288}r^7j^2k - \frac{13}{5760}r^8k^2 - \frac{1}{576}r^7\ell k^2 + \frac{1}{576}r^7jk^2 + \frac{253}{64}r^8\ell - \frac{253}{64}r^8j - \frac{433}{504}r^7\ell j + \frac{433}{504}r^7j^2 + \frac{1}{144}r^6\ell j^2 - \frac{1}{144}r^6j^3 - \frac{57481}{17280}r^8k + \frac{433}{1008}r^7\ell k + \frac{157}{5760}r^6\ell^2k - \frac{359}{1120}r^7jk - \frac{59}{960}r^6\ell jk + \frac{197}{5760}r^6j^2k - \frac{157}{2880}r^7k^2 + \frac{1}{1152}r^6\ell k^2 - \frac{1}{1152}r^6jk^2 + \frac{581}{32}r^8 - \frac{19}{16}r^6\ell^2 - \frac{484691}{60480}r^7j + \frac{19}{8}r^6\ell j + \frac{91}{960}r^5\ell^2j - \frac{719}{720}r^6j^2 - \frac{91}{480}r^5\ell j^2 + \frac{91}{960}r^5j^3 + \frac{484691}{120960}r^7k + \frac{23}{32}r^6\ell k - \frac{91}{1920}r^5\ell^2k - \frac{1}{1152}r^4\ell^3k - \frac{1307}{1440}r^6jk + \frac{253}{2880}r^5\ell jk + \frac{1}{384}r^4\ell^2jk - \frac{233}{5760}r^5j^2k - \frac{1}{384}r^4\ell j^2k + \frac{1}{1152}r^4j^3k + \frac{191}{5760}r^6k^2 + \frac{1}{288}r^5\ell k^2 - \frac{1}{288}r^5jk^2 - \frac{405}{32}r^6\ell + \frac{31}{192}r^4\ell^3 + \frac{405}{32}r^6j + \frac{2977}{1440}r^5\ell j - \frac{31}{64}r^4\ell^2j - \frac{1}{288}r^3\ell^3j - \frac{2977}{1440}r^5j^2 + \frac{271}{576}r^4\ell j^2 + \frac{1}{96}r^3\ell^2j^2 - \frac{85}{576}r^4j^3 - \frac{1}{96}r^3\ell j^3 + \frac{1}{288}r^3j^4 + \frac{675337}{172800}r^6k - \frac{2977}{2880}r^5\ell k - \frac{55}{1152}r^4\ell^2k + \frac{1}{576}r^3\ell^3k + \frac{337}{360}r^5jk + \frac{7}{64}r^4\ell jk - \frac{1}{192}r^3\ell^2jk - \frac{71}{1152}r^4j^2k + \frac{1}{192}r^3\ell j^2k - \frac{1}{576}r^3j^3k + \frac{281}{5760}r^5k^2 - \frac{7}{2304}r^4\ell k^2 + \frac{7}{2304}r^4jk^2 - \frac{473}{16}r^6 + \frac{11}{4}r^4\ell^2 - \frac{1}{96}r^2\ell^4 + \frac{31439}{3456}r^5j - \frac{11}{2}r^4\ell j -$

$\frac{31}{192}r^3\ell^2j + \frac{1}{24}r^2\ell^3j + \frac{1861}{720}r^4j^2 + \frac{31}{96}r^3\ell j^2 - \frac{1}{16}r^2\ell^2j^2 - \frac{31}{192}r^3j^3 + \frac{1}{24}r^2\ell j^3 - \frac{1}{96}r^2j^4 - \frac{31439}{6912}r^5k -$
$\frac{3361}{5760}r^4\ell k + \frac{31}{384}r^3\ell^2k + \frac{1}{1152}r^2\ell^3k + \frac{4313}{5760}r^4jk - \frac{91}{576}r^3\ell jk - \frac{1}{384}r^2\ell^2jk + \frac{89}{1152}r^3j^2k + \frac{1}{384}r^2\ell j^2k -$
$\frac{1}{1152}r^2j^3k - \frac{641}{17280}r^4k^2 - \frac{1}{576}r^3\ell k^2 + \frac{1}{576}r^3jk^2 + \frac{2759}{192}r^4\ell - \frac{47}{192}r^2\ell^3 + \frac{1}{3840}\ell^5 - \frac{2759}{192}r^4j -$
$\frac{155}{96}r^3\ell j + \frac{47}{64}r^2\ell^2j + \frac{1}{288}r\ell^3j - \frac{1}{768}\ell^4j + \frac{155}{96}r^3j^2 - \frac{419}{576}r^2\ell j^2 - \frac{1}{96}r\ell^2j^2 + \frac{1}{384}\ell^3j^2 + \frac{137}{576}r^2j^3 +$
$\frac{1}{96}r\ell j^3 - \frac{1}{384}\ell^2j^3 - \frac{1}{288}rj^4 + \frac{1}{768}\ell j^4 - \frac{1}{3840}j^5 - \frac{12869}{6480}r^4k + \frac{155}{192}r^3\ell k + \frac{59}{2880}r^2\ell^2k - \frac{1}{576}r\ell^3k -$
$\frac{6727}{8640}r^3jk - \frac{23}{480}r^2\ell jk + \frac{1}{192}r\ell^2jk + \frac{79}{2880}r^2j^2k - \frac{1}{192}r\ell j^2k + \frac{1}{576}rj^3k - \frac{31}{2160}r^3k^2 + \frac{1}{576}r^2\ell k^2 -$
$\frac{1}{576}r^2jk^2 + \frac{713}{32}r^4 - \frac{97}{48}r^2\ell^2 + \frac{1}{128}\ell^4 - \frac{402751}{90720}r^3j + \frac{97}{24}r^2\ell j + \frac{1}{15}r\ell^2j - \frac{1}{32}\ell^3j - \frac{1421}{720}r^2j^2 - \frac{2}{15}r\ell j^2 +$
$\frac{3}{64}\ell^2j^2 + \frac{1}{15}rj^3 - \frac{1}{32}\ell j^3 + \frac{1}{128}j^4 + \frac{402751}{181440}r^3k + \frac{155}{1008}r^2\ell k - \frac{1}{30}r\ell^2k - \frac{1013}{5040}r^2jk + \frac{1}{15}r\ell jk -$
$\frac{1}{30}rj^2k + \frac{17}{1440}r^2k^2 - \frac{41}{6}r^2\ell + \frac{17}{192}\ell^3 + \frac{41}{6}r^2j + \frac{128}{315}r\ell j - \frac{17}{64}\ell^2j - \frac{128}{315}rj^2 + \frac{17}{64}\ell j^2 - \frac{17}{192}j^3 +$
$\frac{659}{1800}r^2k - \frac{64}{315}r\ell k + \frac{64}{315}rjk - \frac{311}{40}r^2 + \frac{15}{32}\ell^2 + \frac{983}{1260}rj - \frac{15}{16}\ell j + \frac{15}{32}j^2 - \frac{983}{2520}rk + \frac{137}{120}\ell - \frac{137}{120}j + 1$

$y_6 = -\frac{1}{82944}r^{12}k^3 + \frac{179443}{7257600}r^{12}k^2 - \frac{1}{6912}r^{11}jk^2 + \frac{1}{13824}r^{11}k^3 - \frac{668671847}{239500800}r^{12}k + \frac{49127}{302400}r^{11}jk -$
$\frac{1}{1728}r^{10}j^2k - \frac{49127}{604800}r^{11}k^2 - \frac{7}{2160}r^{10}\ell k^2 + \frac{11}{2880}r^{10}jk^2 - \frac{1}{9216}r^{10}k^3 + \frac{11767}{1152}r^{12} - \frac{5344573}{831600}r^{11}j +$
$\frac{157681}{604800}r^{10}j^2 - \frac{1}{1296}r^9j^3 + \frac{5344573}{1663200}r^{11}k + \frac{6285409}{7257600}r^{10}\ell k - \frac{8177581}{7257600}r^{10}jk - \frac{407}{17280}r^9\ell jk + \frac{427}{17280}r^9j^2k -$
$\frac{140407}{7257600}r^{10}k^2 + \frac{407}{34560}r^9\ell k^2 + \frac{1}{9216}r^8\ell^2k^2 - \frac{103}{8640}r^9jk^2 - \frac{1}{4608}r^8\ell jk^2 + \frac{1}{9216}r^8j^2k^2 - \frac{5}{41472}r^9k^3 -$
$\frac{8747}{960}r^{10}\ell + \frac{8747}{960}r^{10}j + \frac{1706143}{725760}r^9\ell j - \frac{1706143}{725760}r^9j^2 - \frac{61}{1440}r^8\ell j^2 + \frac{61}{1440}r^8j^3 + \frac{229559233}{21772800}r^{10}k -$
$\frac{1706143}{1451520}r^9\ell k - \frac{16469}{161280}r^8\ell^2k + \frac{915379}{1451520}r^9jk + \frac{3977}{16128}r^8\ell jk + \frac{1}{1152}r^7\ell^2jk - \frac{23021}{161280}r^8j^2k - \frac{1}{576}r^7\ell j^2k +$
$\frac{1}{1152}r^7j^3k + \frac{65897}{241920}r^9k^2 - \frac{41}{23040}r^8\ell k^2 - \frac{1}{2304}r^7\ell^2k^2 + \frac{1}{23040}r^8jk^2 + \frac{1}{1152}r^7\ell jk^2 - \frac{1}{2304}r^7j^2k^2 +$
$\frac{11}{27648}r^8k^3 - \frac{93823}{1920}r^{10} + \frac{2111}{768}r^8\ell^2 + \frac{1221817}{51840}r^9j - \frac{2111}{384}r^8\ell j - \frac{6451}{20160}r^7\ell^2j + \frac{152819}{80640}r^8j^2 + \frac{6451}{10080}r^7\ell j^2 +$
$\frac{1}{576}r^6\ell^2j^2 - \frac{19213}{60480}r^7j^3 - \frac{1}{288}r^6\ell j^3 + \frac{1}{576}r^6j^4 - \frac{1221817}{103680}r^9k - \frac{324797}{120960}r^8\ell k + \frac{6451}{40320}r^7\ell^2k +$
$\frac{187}{34560}r^6\ell^3k + \frac{428051}{120960}r^8jk - \frac{5177}{20160}r^7\ell jk - \frac{23}{1280}r^6\ell^2jk + \frac{3763}{40320}r^7j^2k + \frac{227}{11520}r^6\ell j^2k - \frac{247}{34560}r^6j^3k -$
$\frac{257557}{2419200}r^8k^2 - \frac{91}{2880}r^7\ell k^2 + \frac{1}{4608}r^6\ell^2k^2 + \frac{379}{11520}r^7jk^2 - \frac{1}{2304}r^6\ell jk^2 + \frac{1}{4608}r^6j^2k^2 - \frac{1}{13824}r^7k^3 +$
$\frac{4323}{128}r^8\ell - \frac{233}{576}r^6\ell^3 - \frac{4323}{128}r^8j - \frac{852911}{120960}r^7\ell j + \frac{233}{192}r^6\ell^2j + \frac{37}{1920}r^5\ell^3j + \frac{852911}{120960}r^7j^2 - \frac{3173}{2880}r^6\ell j^2 -$
$\frac{37}{640}r^5\ell^2j^2 + \frac{281}{960}r^6j^3 + \frac{37}{640}r^5\ell j^3 - \frac{37}{1920}r^5j^4 - \frac{16123691}{1036800}r^8k + \frac{852911}{241920}r^7\ell k + \frac{2837}{11520}r^6\ell^2k - \frac{37}{3840}r^5\ell^3k -$
$\frac{1}{9216}r^4\ell^4k - \frac{3449251}{1209600}r^7jk - \frac{3481}{5760}r^6\ell jk + \frac{313}{11520}r^5\ell^2jk + \frac{1}{2304}r^4\ell^3jk + \frac{821}{2304}r^6j^2k - \frac{293}{11520}r^5\ell j^2k -$
$\frac{1}{1536}r^4\ell^2j^2k + \frac{91}{11520}r^5j^3k + \frac{1}{2304}r^4\ell j^3k - \frac{1}{9216}r^4j^4k - \frac{4853}{14400}r^7k^2 + \frac{77}{3840}r^6\ell k^2 + \frac{1}{1152}r^5\ell^2k^2 -$
$\frac{211}{11520}r^6jk^2 - \frac{1}{576}r^5\ell jk^2 + \frac{1}{1152}r^5j^2k^2 - \frac{35}{82944}r^6k^3 + \frac{105277}{1152}r^8 - \frac{3043}{384}r^6\ell^2 + \frac{49}{1536}r^4\ell^4 - \frac{10169191}{302400}r^7j +$
$\frac{3043}{192}r^6\ell j + \frac{215}{288}r^5\ell^2j - \frac{49}{384}r^4\ell^3j - \frac{1}{2304}r^3\ell^4j - \frac{49663}{7200}r^6j^2 - \frac{215}{144}r^5\ell j^2 + \frac{433}{2304}r^4\ell^2j^2 + \frac{1}{576}r^3\ell^3j^2 +$
$\frac{643}{864}r^5j^3 - \frac{139}{1152}r^4\ell j^3 - \frac{1}{384}r^3\ell^2j^3 + \frac{131}{4608}r^4j^4 + \frac{1}{576}r^3\ell j^4 - \frac{1}{2304}r^3j^5 + \frac{10169191}{604800}r^7k + \frac{351299}{115200}r^6\ell k -$
$\frac{215}{576}r^5\ell^2k - \frac{1}{108}r^4\ell^3k + \frac{1}{4608}r^3\ell^4k - \frac{469591}{115200}r^6jk + \frac{3979}{5760}r^5\ell jk + \frac{1}{32}r^4\ell^2jk - \frac{1}{1152}r^3\ell^3jk - \frac{201}{640}r^5j^2k -$
$\frac{5}{144}r^4\ell j^2k + \frac{1}{768}r^3\ell^2j^2k + \frac{11}{864}r^4j^3k - \frac{1}{1152}r^3\ell j^3k + \frac{1}{4608}r^3j^4k + \frac{1433399}{7257600}r^6k^2 + \frac{107}{3840}r^5\ell k^2 -$

$$\frac{7}{9216}r^4\ell^2k^2-\frac{509}{17280}r^5jk^2+\frac{7}{4608}r^4\ell jk^2-\frac{7}{9216}r^4j^2k^2+\frac{1}{4608}r^5k^3-\frac{430}{9}r^6\ell+\frac{331}{384}r^4\ell^3-\frac{1}{768}r^2\ell^5+$$
$$\frac{430}{9}r^6j+\frac{266539}{34560}r^5\ell j-\frac{331}{128}r^4\ell^2j-\frac{37}{1152}r^3\ell^3j+\frac{5}{768}r^2\ell^4j-\frac{266539}{34560}r^5j^2+\frac{14339}{5760}r^4\ell j^2+\frac{37}{384}r^3\ell^2j^2-$$
$$\frac{5}{384}r^2\ell^3j^2-\frac{4409}{5760}r^4j^3-\frac{37}{384}r^3\ell j^3+\frac{5}{384}r^2\ell^2j^3+\frac{37}{1152}r^3j^4-\frac{5}{768}r^2\ell j^4+\frac{1}{768}r^2j^5+\frac{242908639}{21772800}r^6k-$$
$$\frac{266539}{69120}r^5\ell k-\frac{4453}{23040}r^4\ell^2k+\frac{37}{2304}r^3\ell^3k+\frac{1}{9216}r^2\ell^4k+\frac{1689881}{483840}r^5jk+\frac{371}{768}r^4\ell jk-\frac{109}{2304}r^3\ell^2jk-$$
$$\frac{1}{2304}r^2\ell^3jk-\frac{19991}{69120}r^4j^2k+\frac{107}{2304}r^3\ell j^2k+\frac{1}{1536}r^2\ell^2j^2k-\frac{35}{2304}r^3j^3k-\frac{1}{2304}r^2\ell j^3k+\frac{1}{9216}r^2j^4k+$$
$$\frac{43973}{241920}r^5k^2-\frac{1507}{69120}r^4\ell k^2-\frac{1}{2304}r^3\ell^2k^2+\frac{163}{7680}r^4jk^2+\frac{1}{1152}r^3\ell jk^2-\frac{1}{2304}r^3j^2k^2+\frac{1}{6912}r^4k^3-$$
$$\frac{32905}{384}r^6+\frac{6305}{768}r^4\ell^2-\frac{23}{512}r^2\ell^4+\frac{1}{46080}\ell^6+\frac{312181}{13440}r^5j-\frac{6305}{384}r^4\ell j-\frac{811}{1440}r^3\ell^2j+\frac{23}{128}r^2\ell^3j+\frac{1}{2304}r\ell^4j-$$
$$\frac{1}{7680}\ell^5j+\frac{1856483}{241920}r^4j^2+\frac{811}{720}r^3\ell j^2-\frac{617}{2304}r^2\ell^2j^2-\frac{1}{576}r\ell^3j^2+\frac{1}{3072}\ell^4j^2-\frac{7289}{12960}r^3j^3+\frac{203}{1152}r^2\ell j^3+$$
$$\frac{1}{384}r\ell^2j^3-\frac{1}{2304}\ell^3j^3-\frac{199}{4608}r^2j^4-\frac{1}{576}r\ell j^4+\frac{1}{3072}\ell^2j^4+\frac{1}{2304}rj^5-\frac{1}{7680}\ell j^5+\frac{1}{46080}j^6-\frac{312181}{26880}r^5k-$$
$$\frac{1086553}{725760}r^4\ell k+\frac{811}{2880}r^3\ell^2k+\frac{133}{34560}r^2\ell^3k-\frac{1}{4608}r\ell^4k+\frac{1475329}{725760}r^4jk-\frac{4727}{8640}r^3\ell jk-\frac{17}{1280}r^2\ell^2jk+$$
$$\frac{1}{1152}r\ell^3jk+\frac{571}{2160}r^3j^2k+\frac{173}{11520}r^2\ell j^2k-\frac{1}{768}r\ell^2j^2k-\frac{193}{34560}r^2j^3k+\frac{1}{1152}r\ell j^3k-\frac{1}{4608}rj^4k-$$
$$\frac{221093}{1814400}r^4k^2-\frac{139}{17280}r^3\ell k^2+\frac{1}{2304}r^2\ell^2k^2+\frac{149}{17280}r^3jk^2-\frac{1}{1152}r^2\ell jk^2+\frac{1}{2304}r^2j^2k^2-\frac{1}{10368}r^3k^3+$$
$$\frac{12341}{384}r^4\ell-\frac{677}{1152}r^2\ell^3+\frac{7}{7680}\ell^5-\frac{12341}{384}r^4j-\frac{658171}{181440}r^3\ell j+\frac{677}{384}r^2\ell^2j+\frac{37}{2880}r\ell^3j-\frac{7}{1536}\ell^4j+$$
$$\frac{658171}{181440}r^3j^2-\frac{1111}{640}r^2\ell j^2-\frac{37}{960}r\ell^2j^2+\frac{7}{768}\ell^3j^2+\frac{3229}{5760}r^2j^3+\frac{37}{960}r\ell j^3-\frac{7}{768}\ell^2j^3-\frac{37}{2880}rj^4+\frac{7}{1536}\ell j^4-$$
$$\frac{7}{7680}j^5-\frac{21092353}{5443200}r^4k+\frac{658171}{362880}r^3\ell k+\frac{3961}{80640}r^2\ell^2k-\frac{37}{5760}r\ell^3k-\frac{3160523}{1814400}r^3jk-\frac{5053}{40320}r^2\ell jk+$$
$$\frac{37}{1920}r\ell^2jk+\frac{1229}{16128}r^2j^2k-\frac{37}{1920}r\ell j^2k+\frac{37}{5760}rj^3k-\frac{10861}{302400}r^3k^2+\frac{13}{1920}r^2\ell k^2-\frac{13}{1920}r^2jk^2+\frac{3055}{72}r^4-$$
$$\frac{1385}{384}r^2\ell^2+\frac{35}{2304}\ell^4-\frac{500549}{64800}r^3j+\frac{1385}{192}r^2\ell j+\frac{551}{4032}r\ell^2j-\frac{35}{576}\ell^3j-\frac{706613}{201600}r^2j^2-\frac{551}{2016}r\ell j^2+\frac{35}{384}\ell^2j^2+$$
$$\frac{551}{4032}rj^3-\frac{35}{576}\ell j^3+\frac{35}{2304}j^4+\frac{500549}{129600}r^3k+\frac{80671}{302400}r^2\ell k-\frac{551}{8064}r\ell^2k-\frac{111439}{302400}r^2jk+\frac{551}{4032}r\ell jk-$$
$$\frac{551}{8064}rj^2k+\frac{641}{25200}r^2k^2-\frac{3683}{360}r^2\ell+\frac{49}{384}\ell^3+\frac{3683}{360}r^2j+\frac{517}{840}r\ell j-\frac{49}{128}\ell^2j-\frac{517}{840}rj^2+\frac{49}{128}\ell j^2-\frac{49}{384}j^3+$$
$$\frac{15397}{29700}r^2k-\frac{517}{1680}r\ell k+\frac{517}{1680}rjk-\frac{419}{40}r^2+\frac{203}{360}\ell^2+\frac{4541}{4620}rj-\frac{203}{180}\ell j+\frac{203}{360}j^2-\frac{4541}{9240}rk+\frac{49}{40}\ell-\frac{49}{40}j+1$$

$$y_7=\frac{97}{829440}r^{14}k^3-\frac{8062423}{87091200}r^{14}k^2+\frac{541}{414720}r^{13}jk^2-\frac{541}{829440}r^{13}k^3-\frac{1}{165888}r^{12}\ell k^3+\frac{1}{165888}r^{12}jk^3+$$
$$\frac{162718124993}{21794572800}r^{14}k-\frac{24737791}{43545600}r^{13}jk+\frac{25}{5184}r^{12}j^2k+\frac{24737791}{87091200}r^{13}k^2+\frac{229633}{14515200}r^{12}\ell k^2-\frac{299633}{14515200}r^{12}jk^2-$$
$$\frac{1}{13824}r^{11}\ell jk^2+\frac{1}{13824}r^{11}j^2k^2+\frac{641}{829440}r^{12}k^3+\frac{1}{27648}r^{11}\ell k^3-\frac{1}{27648}r^{11}jk^3-\frac{43079}{1680}r^{14}+$$
$$\frac{186558023}{11531520}r^{13}j-\frac{9208019}{10886400}r^{12}j^2+\frac{17}{2880}r^{11}j^3-\frac{186558023}{23063040}r^{13}k-\frac{1192117781}{479001600}r^{12}\ell k+\frac{59158171}{17740800}r^{12}jk+$$
$$\frac{1193}{11200}r^{11}\ell jk-\frac{7753}{67200}r^{11}j^2k-\frac{1}{3456}r^{10}\ell j^2k+\frac{1}{3456}r^{10}j^3k+\frac{4709513}{29030400}r^{12}k^2-\frac{1193}{22400}r^{11}\ell k^2-$$
$$\frac{127}{138240}r^{10}\ell^2k^2+\frac{767779}{14515200}r^{11}jk^2+\frac{49}{23040}r^{10}\ell jk^2-\frac{167}{138240}r^{10}j^2k^2+\frac{275}{165888}r^{11}k^3-\frac{1}{18432}r^{10}\ell k^3+$$
$$\frac{1}{18432}r^{10}jk^3+\frac{248957}{11520}r^{12}\ell-\frac{248957}{11520}r^{12}j-\frac{250869097}{39916800}r^{11}\ell j+\frac{250869097}{39916800}r^{11}j^2+\frac{213121}{1209600}r^{10}\ell j^2-$$
$$\frac{213121}{1209600}r^{10}j^3-\frac{1}{2592}r^9\ell j^3+\frac{1}{2592}r^9j^4-\frac{429892039}{13305600}r^{12}k+\frac{250869097}{79833600}r^{11}\ell k+\frac{9746269}{29030400}r^{10}\ell^2k-$$
$$\frac{430178059}{479001600}r^{11}jk-\frac{12303721}{14515200}r^{10}\ell jk-\frac{467}{69120}r^9\ell^2jk+\frac{14351573}{29030400}r^{10}j^2k+\frac{487}{34560}r^9\ell j^2k-\frac{169}{23040}r^9j^3k-$$
$$\frac{97730593}{87091200}r^{11}k^2-\frac{134947}{14515200}r^{10}\ell k^2+\frac{467}{138240}r^9\ell^2k^2+\frac{1}{55296}r^8\ell^3k^2+\frac{389747}{14515200}r^{10}jk^2-\frac{59}{8640}r^9\ell jk^2-$$

$$\tfrac{1}{18432}r^8\ell^2jk^2+\tfrac{53}{15360}r^9j^2k^2+\tfrac{1}{18432}r^8\ell j^2k^2-\tfrac{1}{55296}r^8j^3k^2-\tfrac{629}{165888}r^{10}k^3-\tfrac{5}{82944}r^9\ell k^3+$$
$$\tfrac{5}{82944}r^9jk^3+\tfrac{153181}{1152}r^{12}-\tfrac{12523}{1920}r^{10}\ell^2-\tfrac{43118927}{633600}r^{11}j+\tfrac{12523}{960}r^{10}\ell j+\tfrac{401857}{414720}r^9\ell^2j-\tfrac{35561539}{10886400}r^{10}j^2-$$
$$\tfrac{401857}{207360}r^9\ell j^2-\tfrac{71}{5760}r^8\ell^2j^2+\tfrac{393041}{414720}r^9j^3+\tfrac{71}{2880}r^8\ell j^3-\tfrac{71}{5760}r^8j^4+\tfrac{43118927}{1267200}r^{11}k+\tfrac{4985291}{544320}r^{10}\ell k-$$
$$\tfrac{401857}{829440}r^9\ell^2k-\tfrac{1475}{64512}r^8\ell^3k-\tfrac{45049897}{3628800}r^{10}jk+\tfrac{1789471}{2903040}r^9\ell jk+\tfrac{26101}{322560}r^8\ell^2jk+\tfrac{1}{6912}r^7\ell^3jk-$$
$$\tfrac{580807}{5806080}r^9j^2k-\tfrac{29797}{322560}r^8\ell j^2k-\tfrac{1}{2304}r^7\ell^2j^2k+\tfrac{11071}{322560}r^8j^3k+\tfrac{1}{2304}r^7\ell j^3k-\tfrac{1}{6912}r^7j^4k+$$
$$\tfrac{19150633}{87091200}r^{10}k^2+\tfrac{42647}{241920}r^9\ell k^2-\tfrac{7}{11520}r^8\ell^2k^2-\tfrac{1}{13824}r^7\ell^3k^2-\tfrac{179681}{967680}r^9jk^2+\tfrac{1}{2880}r^8\ell jk^2+$$
$$\tfrac{1}{4608}r^7\ell^2jk^2+\tfrac{1}{3840}r^8j^2k^2-\tfrac{1}{4608}r^7\ell j^2k^2+\tfrac{1}{13824}r^7j^3k^2-\tfrac{511}{829440}r^9k^3+\tfrac{11}{55296}r^8\ell k^3-\tfrac{11}{55296}r^8jk^3-$$
$$\tfrac{349141}{3840}r^{10}\ell+\tfrac{1549}{1536}r^8\ell^3+\tfrac{349141}{3840}r^{10}j+\tfrac{10837363}{483840}r^9\ell j-\tfrac{1549}{512}r^8\ell^2j-\tfrac{281}{3780}r^7\ell^3j-\tfrac{10837363}{483840}r^9j^2+$$
$$\tfrac{395971}{161280}r^8\ell j^2+\tfrac{281}{1260}r^7\ell^2j^2+\tfrac{1}{3456}r^6\ell^3j^2-\tfrac{70681}{161280}r^8j^3-\tfrac{6709}{30240}r^7\ell j^3-\tfrac{1}{1152}r^6\ell^2j^3+\tfrac{2213}{30240}r^7j^4+$$
$$\tfrac{1}{1152}r^6\ell j^4-\tfrac{1}{3456}r^6j^5+\tfrac{619372423}{10886400}r^{10}k-\tfrac{10837363}{967680}r^9\ell k-\tfrac{981997}{967680}r^8\ell^2k+\tfrac{281}{7560}r^7\ell^3k+\tfrac{217}{276480}r^6\ell^4k+$$
$$\tfrac{8000951}{1036800}r^9jk+\tfrac{1257889}{483840}r^8\ell jk-\tfrac{7543}{80640}r^7\ell^2jk-\tfrac{79}{23040}r^6\ell^3jk-\tfrac{167869}{107520}r^8j^2k+\tfrac{2977}{40320}r^7\ell j^2k+$$
$$\tfrac{257}{46080}r^6\ell^2j^2k-\tfrac{845}{48384}r^7j^3k-\tfrac{277}{69120}r^6\ell j^3k+\tfrac{11}{10240}r^6j^4k+\tfrac{50547131}{29030400}r^9k^2-\tfrac{122719}{1612800}r^8\ell k^2-$$
$$\tfrac{23}{2560}r^7\ell^2k^2+\tfrac{1}{27648}r^6\ell^3k^2+\tfrac{253357}{4838400}r^8jk^2+\tfrac{143}{7680}r^7\ell jk^2-\tfrac{1}{9216}r^6\ell^2jk^2-\tfrac{37}{3840}r^7j^2k^2+\tfrac{1}{9216}r^6\ell j^2k^2-$$
$$\tfrac{1}{27648}r^6j^3k^2+\tfrac{4619}{829440}r^8k^3-\tfrac{1}{27648}r^7\ell k^3+\tfrac{1}{27648}r^7jk^3-\tfrac{179663}{640}r^{10}+\tfrac{8585}{384}r^8\ell^2-\tfrac{103}{1152}r^6\ell^4+$$
$$\tfrac{84270617}{725760}r^9j-\tfrac{8585}{192}r^8\ell j-\tfrac{274063}{96768}r^7\ell^2j+\tfrac{103}{288}r^6\ell^3j+\tfrac{131}{46080}r^5\ell^4j+\tfrac{63328991}{3628800}r^8j^2+\tfrac{274063}{48384}r^7\ell j^2-$$
$$\tfrac{121}{240}r^6\ell^2j^2-\tfrac{131}{11520}r^5\ell^3j^2-\tfrac{271319}{96768}r^7j^3+\tfrac{211}{720}r^6\ell j^3+\tfrac{131}{7680}r^5\ell^2j^3-\tfrac{329}{5760}r^6j^4-\tfrac{131}{11520}r^5\ell j^4+$$
$$\tfrac{131}{46080}r^5j^5-\tfrac{84270617}{1451520}r^9k-\tfrac{191096141}{14515200}r^8\ell k+\tfrac{274063}{193536}r^7\ell^2k+\tfrac{931}{17280}r^6\ell^3k-\tfrac{131}{92160}r^5\ell^4k-\tfrac{1}{92160}r^4\ell^5k+$$
$$\tfrac{87431059}{4838400}r^8jk-\tfrac{1937237}{806400}r^7\ell jk-\tfrac{1117}{5760}r^6\ell^2jk+\tfrac{373}{69120}r^5\ell^3jk+\tfrac{1}{18432}r^4\ell^4jk+\tfrac{4566047}{4838400}r^7j^2k+$$
$$\tfrac{649}{2880}r^6\ell j^2k-\tfrac{353}{46080}r^5\ell^2j^2k-\tfrac{1}{9216}r^4\ell^3j^2k-\tfrac{737}{8640}r^6j^3k+\tfrac{37}{7680}r^5\ell j^3k+\tfrac{1}{9216}r^4\ell^2j^3k-\tfrac{313}{276480}r^5j^4k-$$
$$\tfrac{1}{18432}r^4\ell j^4k+\tfrac{1}{92160}r^4j^5k-\tfrac{9485129}{12441600}r^8k^2-\tfrac{18569}{86400}r^7\ell k^2+\tfrac{271}{46080}r^6\ell^2k^2+\tfrac{1}{6912}r^5\ell^3k^2+\tfrac{481591}{2073600}r^7jk^2-$$
$$\tfrac{251}{23040}r^6\ell jk^2-\tfrac{1}{2304}r^5\ell^2jk^2+\tfrac{77}{15360}r^6j^2k^2+\tfrac{1}{2304}r^5\ell j^2k^2-\tfrac{1}{6912}r^5j^3k^2-\tfrac{1307}{829440}r^7k^3-\tfrac{35}{165888}r^6\ell k^3+$$
$$\tfrac{35}{165888}r^6jk^3+\tfrac{352609}{2304}r^8\ell-\tfrac{6295}{2304}r^6\ell^3+\tfrac{71}{15360}r^4\ell^5-\tfrac{352609}{2304}r^8j-\tfrac{18791891}{604800}r^7\ell j+\tfrac{6295}{768}r^6\ell^2j+$$
$$\tfrac{5863}{34560}r^5\ell^3j-\tfrac{71}{3072}r^4\ell^4j-\tfrac{1}{23040}r^3\ell^5j+\tfrac{18791891}{604800}r^7j^2-\tfrac{162427}{21600}r^6\ell j^2-\tfrac{5863}{11520}r^5\ell^2j^2+\tfrac{631}{13824}r^4\ell^3j^2+$$
$$\tfrac{1}{4608}r^3\ell^4j^2+\tfrac{177583}{86400}r^6j^3+\tfrac{17549}{34560}r^5\ell j^3-\tfrac{205}{4608}r^4\ell^2j^3-\tfrac{1}{2304}r^3\ell^3j^3-\tfrac{647}{3840}r^5j^4+\tfrac{197}{9216}r^4\ell j^4+$$
$$\tfrac{1}{2304}r^3\ell^2j^4-\tfrac{559}{138240}r^4j^5-\tfrac{1}{4608}r^3\ell j^5+\tfrac{1}{23040}r^3j^6-\tfrac{3964220441}{76204800}r^8k+\tfrac{18791891}{1209600}r^7\ell k+\tfrac{1551797}{1382400}r^6\ell^2k-$$
$$\tfrac{5863}{69120}r^5\ell^3k-\tfrac{73}{55296}r^4\ell^4k+\tfrac{1}{46080}r^3\ell^5k-\tfrac{561306737}{43545600}r^7jk-\tfrac{673211}{230400}r^6\ell jk+\tfrac{917}{3840}r^5\ell^2jk+\tfrac{3}{512}r^4\ell^3jk-$$
$$\tfrac{1}{9216}r^3\ell^4jk+\tfrac{7404007}{4147200}r^6j^2k-\tfrac{5101}{23040}r^5\ell j^2k-\tfrac{89}{9216}r^4\ell^2j^2k+\tfrac{1}{4608}r^3\ell^3j^2k+\tfrac{233}{3456}r^5j^3k+\tfrac{97}{13824}r^4\ell j^3k-$$
$$\tfrac{1}{4608}r^3\ell^2j^3k-\tfrac{35}{18432}r^4j^4k+\tfrac{1}{9216}r^3\ell j^4k-\tfrac{1}{46080}r^3j^5k-\tfrac{115201339}{87091200}r^7k^2+\tfrac{1931099}{14515200}r^6\ell k^2+$$
$$\tfrac{361}{46080}r^5\ell^2k^2-\tfrac{7}{55296}r^4\ell^3k^2-\tfrac{1726699}{14515200}r^6jk^2-\tfrac{569}{34560}r^5\ell jk^2+\tfrac{7}{18432}r^4\ell^2jk^2+\tfrac{1193}{138240}r^5j^2k^2-$$
$$\tfrac{7}{18432}r^4\ell j^2k^2+\tfrac{7}{55296}r^4j^3k^2-\tfrac{713}{207360}r^6k^3+\tfrac{1}{9216}r^5\ell k^3-\tfrac{1}{9216}r^5jk^3+\tfrac{360319}{1152}r^8-\tfrac{34007}{1152}r^6\ell^2+$$
$$\tfrac{277}{1536}r^4\ell^4-\tfrac{1}{7680}r^2\ell^6-\tfrac{17709193}{172800}r^7j+\tfrac{34007}{576}r^6\ell j+\tfrac{83387}{27648}r^5\ell^2j-\tfrac{277}{384}r^4\ell^3j-\tfrac{43}{9216}r^3\ell^4j+\tfrac{1}{1280}r^2\ell^5j-$$

$$\tfrac{282137077}{10886400}r^6j^2 - \tfrac{83387}{13824}r^5\ell j^2 + \tfrac{4049}{3840}r^4\ell^2j^2 + \tfrac{43}{2304}r^3\ell^3j^2 - \tfrac{1}{512}r^2\ell^4j^2 + \tfrac{414647}{138240}r^5j^3 - \tfrac{1279}{1920}r^4\ell j^3 -$$
$$\tfrac{43}{1536}r^3\ell^2j^3 + \tfrac{1}{384}r^2\ell^3j^3 + \tfrac{391}{2560}r^4j^4 + \tfrac{43}{2304}r^3\ell j^4 - \tfrac{1}{512}r^2\ell^2j^4 - \tfrac{43}{9216}r^3j^5 + \tfrac{1}{1280}r^2\ell j^5 - \tfrac{1}{7680}r^2j^6 +$$
$$\tfrac{17709193}{345600}r^7k + \tfrac{8020637}{870912}r^6\ell k - \tfrac{83387}{55296}r^5\ell^2k - \tfrac{1139}{27648}r^4\ell^3k + \tfrac{43}{18432}r^3\ell^4k + \tfrac{1}{92160}r^2\ell^5k - \tfrac{10332373}{806400}r^6jk +$$
$$\tfrac{2697881}{967680}r^5\ell jk + \tfrac{6967}{46080}r^4\ell^2jk - \tfrac{127}{13824}r^3\ell^3jk - \tfrac{1}{18432}r^2\ell^4jk - \tfrac{809723}{645120}r^5j^2k - \tfrac{24677}{138240}r^4\ell j^2k +$$
$$\tfrac{125}{9216}r^3\ell^2j^2k + \tfrac{1}{9216}r^2\ell^3j^2k + \tfrac{3157}{46080}r^4j^3k - \tfrac{41}{4608}r^3\ell j^3k - \tfrac{1}{9216}r^2\ell^2j^3k + \tfrac{121}{55296}r^3j^4k + \tfrac{1}{18432}r^2\ell j^4k -$$
$$\tfrac{1}{92160}r^2j^5k + \tfrac{3500221}{4838400}r^6k^2 + \tfrac{27583}{241920}r^5\ell k^2 - \tfrac{433}{69120}r^4\ell^2k^2 - \tfrac{1}{13824}r^3\ell^3k^2 - \tfrac{91121}{725760}r^5jk^2 + \tfrac{47}{3840}r^4\ell jk^2 +$$
$$\tfrac{1}{4608}r^3\ell^2jk^2 - \tfrac{413}{69120}r^4j^2k^2 - \tfrac{1}{4608}r^3\ell j^2k^2 + \tfrac{1}{13824}r^3j^3k^2 + \tfrac{169}{103680}r^5k^3 + \tfrac{1}{13824}r^4\ell k^3 - \tfrac{1}{13824}r^4jk^3 -$$
$$\tfrac{303053}{2304}r^6\ell + \tfrac{4111}{1536}r^4\ell^3 - \tfrac{19}{3072}r^2\ell^5 + \tfrac{1}{645120}\ell^7 + \tfrac{303053}{2304}r^6j + \tfrac{6064705}{290304}r^5\ell j - \tfrac{4111}{512}r^4\ell^2j - \tfrac{4313}{34560}r^3\ell^3j +$$
$$\tfrac{95}{3072}r^2\ell^4j + \tfrac{1}{23040}r\ell^5j - \tfrac{1}{92160}\ell^6j - \tfrac{6064705}{290304}r^5j^2 + \tfrac{1238869}{161280}r^4\ell j^2 + \tfrac{4313}{11520}r^3\ell^2j^2 - \tfrac{851}{13824}r^2\ell^3j^2 -$$
$$\tfrac{1}{4608}r\ell^4j^2 + \tfrac{1}{30720}\ell^5j^2 - \tfrac{375559}{161280}r^4j^3 - \tfrac{38777}{103680}r^3\ell j^3 + \tfrac{281}{4608}r^2\ell^2j^3 + \tfrac{1}{2304}r\ell^3j^3 - \tfrac{1}{18432}\ell^4j^3 +$$
$$\tfrac{12899}{103680}r^3j^4 - \tfrac{277}{9216}r^2\ell j^4 - \tfrac{1}{2304}r\ell^2j^4 + \tfrac{1}{18432}\ell^3j^4 + \tfrac{163}{27648}r^2j^5 + \tfrac{1}{4608}r\ell j^5 - \tfrac{1}{30720}\ell^2j^5 - \tfrac{1}{23040}rj^6 +$$
$$\tfrac{1}{92160}\ell j^6 - \tfrac{1}{645120}j^7 + \tfrac{188357023}{7257600}r^6k - \tfrac{6064705}{580608}r^5\ell k - \tfrac{97369}{181440}r^4\ell^2k + \tfrac{4313}{69120}r^3\ell^3k + \tfrac{37}{69120}r^2\ell^4k -$$
$$\tfrac{1}{46080}r\ell^5k + \tfrac{412140959}{43545600}r^5jk + \tfrac{128923}{90720}r^4\ell jk - \tfrac{12631}{69120}r^3\ell^2jk - \tfrac{7}{2880}r^2\ell^3jk + \tfrac{1}{9216}r\ell^4jk - \tfrac{159917}{181440}r^4j^2k +$$
$$\tfrac{12283}{69120}r^3\ell j^2k + \tfrac{47}{11520}r^2\ell^2j^2k - \tfrac{1}{4608}r\ell^3j^2k - \tfrac{793}{13824}r^3j^3k - \tfrac{13}{4320}r^2\ell j^3k + \tfrac{1}{4608}r\ell^2j^3k + \tfrac{19}{23040}r^2j^4k -$$
$$\tfrac{1}{9216}r\ell j^4k + \tfrac{1}{46080}rj^5k + \tfrac{10677979}{21772800}r^5k^2 - \tfrac{578581}{7257600}r^4\ell k^2 - \tfrac{77}{34560}r^3\ell^2k^2 + \tfrac{1}{13824}r^2\ell^3k^2 + \tfrac{556181}{7257600}r^4jk^2 +$$
$$\tfrac{41}{8640}r^3\ell jk^2 - \tfrac{1}{4608}r^2\ell^2jk^2 - \tfrac{29}{11520}r^3j^2k^2 + \tfrac{1}{4608}r^2\ell j^2k^2 - \tfrac{1}{13824}r^2j^3k^2 + \tfrac{1}{1296}r^4k^3 - \tfrac{1}{20736}r^3\ell k^3 +$$
$$\tfrac{1}{20736}r^3jk^3 - \tfrac{380639}{1920}r^6 + \tfrac{1197}{64}r^4\ell^2 - \tfrac{539}{4608}r^2\ell^4 + \tfrac{1}{11520}\ell^6 + \tfrac{849163}{17280}r^5j - \tfrac{1197}{32}r^4\ell j - \tfrac{14323}{10368}r^3\ell^2j +$$
$$\tfrac{539}{1152}r^2\ell^3j + \tfrac{7}{3840}r\ell^4j - \tfrac{1}{1920}\ell^5j + \tfrac{23701187}{1360800}r^4j^2 + \tfrac{14323}{5184}r^3\ell j^2 - \tfrac{7997}{11520}r^2\ell^2j^2 - \tfrac{7}{960}r\ell^3j^2 + \tfrac{1}{768}\ell^4j^2 -$$
$$\tfrac{71431}{51840}r^3j^3 + \tfrac{869}{1920}r^2\ell j^3 + \tfrac{7}{640}r\ell^2j^3 - \tfrac{1}{576}\ell^3j^3 - \tfrac{2519}{23040}r^2j^4 - \tfrac{7}{960}r\ell j^4 + \tfrac{1}{768}\ell^2j^4 + \tfrac{7}{3840}rj^5 - \tfrac{1}{1920}\ell j^5 +$$
$$\tfrac{1}{11520}j^6 - \tfrac{849163}{34560}r^5k - \tfrac{33999949}{10886400}r^4\ell k + \tfrac{14323}{20736}r^3\ell^2k + \tfrac{4927}{483840}r^2\ell^3k - \tfrac{7}{7680}r\ell^4k + \tfrac{16000051}{3628800}r^4jk -$$
$$\tfrac{2426239}{1814400}r^3\ell jk - \tfrac{2053}{53760}r^2\ell^2jk + \tfrac{7}{1920}r\ell^3jk + \tfrac{2326633}{3628800}r^3j^2k + \tfrac{7391}{161280}r^2\ell j^2k - \tfrac{7}{1280}r\ell^2j^2k -$$
$$\tfrac{8623}{483840}r^2j^3k + \tfrac{7}{1920}r\ell j^3k - \tfrac{7}{7680}rj^4k - \tfrac{3211367}{10886400}r^4k^2 - \tfrac{13381}{604800}r^3\ell k^2 + \tfrac{11}{5760}r^2\ell^2k^2 + \tfrac{4997}{201600}r^3jk^2 -$$
$$\tfrac{11}{2880}r^2\ell jk^2 + \tfrac{11}{5760}r^2j^2k^2 - \tfrac{23}{51840}r^3k^3 + \tfrac{349303}{5760}r^4\ell - \tfrac{2579}{2304}r^2\ell^3 + \tfrac{23}{11520}\ell^5 - \tfrac{349303}{5760}r^4j - \tfrac{4099397}{604800}r^3\ell j +$$
$$\tfrac{2579}{768}r^2\ell^2j + \tfrac{1189}{40320}r\ell^3j - \tfrac{23}{2304}\ell^4j + \tfrac{4099397}{604800}r^3j^2 - \tfrac{3983309}{1209600}r^2\ell j^2 - \tfrac{1189}{13440}r\ell^2j^2 + \tfrac{23}{1152}\ell^3j^2 +$$
$$\tfrac{1275359}{1209600}r^2j^3 + \tfrac{1189}{13440}r\ell j^3 - \tfrac{23}{1152}\ell^2j^3 - \tfrac{1189}{40320}rj^4 + \tfrac{23}{2304}\ell j^4 - \tfrac{23}{11520}j^5 - \tfrac{399350353}{59875200}r^4k + \tfrac{4099397}{1209600}r^3\ell k +$$
$$\tfrac{56353}{604800}r^2\ell^2k - \tfrac{1189}{80640}r\ell^3k - \tfrac{561187}{172800}r^3jk - \tfrac{76007}{302400}r^2\ell jk + \tfrac{1189}{26880}r\ell^2jk + \tfrac{3543}{22400}r^2j^2k - \tfrac{1189}{26880}r\ell j^2k +$$
$$\tfrac{1189}{80640}rj^3k - \tfrac{10693}{151200}r^3k^2 + \tfrac{9827}{604800}r^2\ell k^2 - \tfrac{9827}{604800}r^2jk^2 + \tfrac{641}{9}r^4 - \tfrac{32609}{5760}r^2\ell^2 + \tfrac{7}{288}\ell^4 - \tfrac{60256927}{4989600}r^3j +$$
$$\tfrac{32609}{2880}r^2\ell j + \tfrac{4609}{20160}r\ell^2j - \tfrac{7}{72}\ell^3j - \tfrac{3316553}{604800}r^2j^2 - \tfrac{4609}{10080}r\ell j^2 + \tfrac{7}{48}\ell^2j^2 + \tfrac{4609}{20160}rj^3 - \tfrac{7}{72}\ell j^3 + \tfrac{7}{288}j^4 +$$
$$\tfrac{60256927}{9979200}r^3k + \tfrac{45343}{110880}r^2\ell k - \tfrac{4609}{40320}r\ell^2k - \tfrac{975473}{1663200}r^2jk + \tfrac{4609}{20160}r\ell jk - \tfrac{4609}{40320}rj^2k + \tfrac{839}{18900}r^2k^2 -$$
$$\tfrac{10187}{720}r^2\ell + \tfrac{967}{5760}\ell^3 + \tfrac{10187}{720}r^2j + \tfrac{23413}{27720}r\ell j - \tfrac{967}{1920}\ell^2j - \tfrac{23413}{27720}rj^2 + \tfrac{967}{1920}\ell j^2 - \tfrac{967}{5760}j^3 + \tfrac{25965067}{37837800}r^2k -$$
$$\tfrac{23413}{55440}r\ell k + \tfrac{23413}{55440}rjk - \tfrac{3739}{280}r^2 + \tfrac{469}{720}\ell^2 + \tfrac{70933}{60060}rj - \tfrac{469}{360}\ell j + \tfrac{469}{720}j^2 - \tfrac{70933}{120120}rk + \tfrac{363}{280}\ell - \tfrac{363}{280}j + 1$$

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